Extension of Planck's law of thermal radiation to systems with a steady heat flux

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Abstract

Planck's law of thermal radiation is limited to equilibrium systems that have a definite temperature and do not carry any heat flux. Here we extend it to steady-state systems with a constant heat flux. The obtained formulas explicitly describe the spectrum of thermal radiation in every direction and provide a sound basis for the self-consistent analysis of radiative heat transport across interfaces, gaps, layered and other important structures.

1 Introduction

At the end of the 1800s the physics community was puzzled by its inability to explain the spectrum of thermal radiation in terms of electrodynamics, thermodynamics and the atomic theory of matter, which had been triumphantly developed in the preceding decades. This failure was especially troublesome and annoying because the phenomenon of thermal radiation appeared to be a simple consequence of well-established physics. Thus, according to the atomic theory matter is composed of particles that have a complicated pattern of electric charges performing perpetual thermal motion. And the laws of electrodynamics imply that accelerating electric charges radiate electromagnetic waves that affect other charged particles. From this it follows that particles may exchange energy even if they never collide, and then according to thermodynamics any completely isolated domain which does not exchange energy with its surroundings, eventually comes to thermodynamical equilibrium, which, in particular, means that the average amount of electromagnetic energy emitted from any sub-domain equals the amount of electromagnetic energy absorbed by this domain. Since all of the involved processes could be described by the laws of physics known by the end of the 1800s, it was anticipated that the spectrum of thermal radiation could be easily computed. However, it soon became clear that the predictions were radically different from the observations.

The contradiction between the theory of thermal radiation and experiments was resolved in 1900 by Planck's celebrated work [1, 2] and the formula

$$\mathcal{E}_t(\omega, T) = \frac{\hbar\omega^3}{\pi^2 c^3 (\mathrm{e}^{\hbar\omega/\kappa T} - 1)}, \tag{1.1}$$

which represents the spectrum of thermal radiation from a perfect absorber ("black body") of electromagnetic waves in thermal equilibrium at temperature T. This formula was mathematically deduced from the properties of entropy established in equilibrium thermodynamics and from Planck's ground-breaking, but at that time unjustified, conjecture that the energy of a harmonic oscillator at angular frequency ω can take only multiple values of $\hbar\omega$, where $\hbar$ was a newly introduced physical constant usually referred to as the reduced Planck constant [1]. A slightly modified version of this conjecture was later confirmed by quantum mechanics, which established that the energy of a harmonic oscillator can take only the values $W_n = \hbar\omega(n + \frac{1}{2})$. As a result, Planck's formula (1.1) appears as a mathematical consequence of electrodynamics, thermodynamics and quantum mechanics, which therefore implies that it may be violated when the assumptions used in its derivation are not met, and that only a complete understanding of its derivation can guarantee that the law is applied correctly.

Thermal radiation is a very common phenomenon observed by everyone on a sunny day, and, at the same time, studied in advanced scientific laboratories. The spectrum of thermal radiation in equilibrium systems is described by a universal Planck's law, which does not depend on particular properties of a considered system. Due to its universality, Planck's law is customarily applied to problems arising in different areas ranging from astrophysics to nanotechnology, but in many cases it is applied to systems that do not satisfy the conditions that guarantee that this law is valid. For example, Planck's law is often used for the computation of heat flux between bodies at different temperatures, but since this law is derived from the assumption of thermodynamical equilibrium it is valid only in systems without heat flux, and, therefore, it can not be legitimately applied to the computation of a non-vanishing heat flux.

It is important to realize that a formally unjustified application of some rule does not inevitably lead to wrong results. For example, the accidental replacement of $\sin(x)$ by $\tan(x)$ may be unnoticed if $|x| \ll 1$, but it may be catastrophic if $x \approx \pi/2$. Similarly, applications of Planck's law to computation of the radiative heat transport between two black bodies maintained at different temperatures may be well justified if the distance between these bodies is sufficiently large, but they become invalid in the cases when this distance is smaller than a few characteristic wavelengths of thermally radiated electromagnetic waves. In particular, as shown below, a straightforward application of Planck's law to the analysis of radiative transport through a gap with vanishing width gives the absurd prediction that an imaginary interface in an unbounded material has a non-vanishing thermal resistance.

One of the main tasks of the theory of radiative heat transport is to provide estimates of the radiative heat flux between separated black bodies A and B maintained at temperatures T_A and T_B and exchanging thermal energy with each other. It is obvious that to get such estimates it suffices to compute the spectra of radiation $\mathcal{E}_{A\to B}(\omega, T_A)$ and $\mathcal{E}_{B\to A}(\omega, T_B)$ from A to B and from B to A, respectively. If these spectra are known then the flux can be computed as

$$Q = \mathbb{K}\big\{\mathcal{E}_{A\to B}(\omega, T_A) - \mathcal{E}_{B\to A}(\omega, T_B)\big\}, \tag{1.2}$$

where $\mathbb{K}$ is some operator that takes into account the speeds and paths of radiation. To use this expression for computing the radiative heat flux Q between A and B it is necessary to find the spectra of radiation $\mathcal{E}_{A\to B}(\omega, T_A)$ and $\mathcal{E}_{B\to A}(\omega, T_B)$, but it is useless to describe them by Planck's law, which is valid only in the case of thermal equilibrium, and when it is *a priori* known that $Q = 0$. Therefore, these spectra have to be computed specifically for the case when there is a

[1] Planck's formula is often written in a different form, which can be obtained from (1.1) by the substitutions $\hbar = h/2\pi$ and $\omega = 2\pi\nu$, where h is the Planck constant and $\nu = \omega/2\pi$ is the frequency representing the "number of cycles per unit of time" instead of the "number of radians per unit of time" represented by the angular frequency ω.

non-vanishing heat flux $Q \neq 0$, and this means that these spectra have the structure

$$\mathcal{E}_{A\to B}(\omega, T_A) \equiv \mathcal{E}_{A\to B}(\omega, T_A, Q), \qquad \mathcal{E}_{B\to A}(\omega, T_B) \equiv \mathcal{E}_{B\to A}(\omega, T_B, Q), \tag{1.3}$$

which reflects the fact that they are functions of the temperatures and of the heat flux. Then, substituting (1.3) into (1.2) we obtain the equation

$$Q = \mathbb{K}\big\{\mathcal{E}_{A\to B}(\omega, T_A, Q) - \mathcal{E}_{B\to A}(\omega, T_B, Q)\big\}, \tag{1.4}$$

which connects T_A, T_B and Q. If the spectra (1.3) and the operator $\mathbb{K}$ are known, then this equation can be used to determine the heat flux Q between two bodies at temperatures T_A and T_B.

So, the outlined approach to the analysis of radiative heat transport between black bodies relies on the availability of an efficient description of the spectra of thermal radiation in non-equilibrium systems with non-vanishing heat flux. Therefore, the computation of such spectra may be regarded as the decisive step towards the implementation of the proposed approach, and the rest of this paper is devoted to this fundamental problem, which essentially extends Planck's classical law to steady-state non-equilibrium systems.

We start with an outline, in Section 2, of the derivation of Planck's classical law with emphasis on the key physical and mathematical assumptions that set the limits of its applicability and play an important role in our subsequent analysis. Then, in Section 3 we formulate the limits of applicability of this law and discuss its meaning, which is often misinterpreted. Although this material is not formally used in the rest of the paper, we believe that it helps to understand the phenomenon of radiative heat transport from the point of view of fundamental laws of physics. Finally, in Section 4 we derive a generalized version of Planck's law that describes the spectrum of thermal radiation in the presence of a steady heat flux, and in Section 5 we discuss consequences of the main results and outline some of their future applications.

2 Planck's law of electromagnetic radiation

Planck's formula was originally derived by using combinatorial analysis of the entropy of an ensemble of harmonic oscillators that were conjectured to have discrete energy levels [1]. This conjecture and its consequences were the first steps toward quantum theory, which was formulated almost three decades later and which made Planck's original reasoning obsolete. Therefore, we find it appropriate to outline the derivation of Planck's law of thermal radiation not following the steps of Planck's original work [1, 2] but rather from the modern point of view adopted in most university physics courses [3, 4, 5].

Assume that vibrating electric charges do not directly interact with each other but instead interact with the local electromagnetic field. Thus, the radiation of energy by a vibrating particle may be considered as the process whereby an oscillator passes some of its energy to the electromagnetic field, and the absorbtion of energy may be considered as the process whereby an oscillator takes energy from the electromagnetic field. From this point of view the energy concentrated in a spatial domain G consists of two forms: the energy of particles located in G and the energy of the electromagnetic field in this domain. If the system is in thermodynamical equilibrium, then each form of energy is itself thermodynamically balanced [3, 4, 5], and the electromagnetic field can be considered as an ensemble of electromagnetic oscillators in thermodynamical equilibrium.

It is well known [3, 4] that the electromagnetic energy is evenly split between fields with two different polarizations, and that each of these fields can be described by the Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0, \qquad k = \omega/c, \tag{2.1}$$

where ω is the angular frequency and c is the speed of electromagnetic waves in the considered material. If the electromagnetic field is localized in some domain G then ϕ must obey certain boundary conditions, and the theory of partial differential equations implies [6] that solutions of the Helmholtz equation (2.1), accompanied by the boundary conditions, can only exist if the frequency ω takes one of the spectral values ω_m, which are determined by the shape of the domain G, by the boundary conditions and by the wave speed c in the material. These frequencies and the corresponding fields ϕ_m are often referred to as eigen-frequencies and eigen-oscillations, respectively.

The theory of partial differential equations implies that any field in G can be represented by a superposition of the eigen-oscillations ϕ_m and that the total energy of this superposition is equal to the sum of the energies of the individual eigen-oscillations. Let $\mathcal{P}^2(\omega, T)$ be the average energy of the eigen-oscillation that has the frequency ω and belongs to the equilibrium ensemble at temperature T, and let $N_G(\omega)$ be the number of spectral frequencies ω_m satisfying the bound $\omega_m < \omega$, so that the difference

$$\mathrm{d}N_G(\omega) = N_G(\omega + \mathrm{d}\omega) - N_G(\omega), \tag{2.2}$$

represents the number of eigen-oscillations at frequency ω_m from the interval $\omega \le \omega_m < \omega + \mathrm{d}\omega$. Then, the average energy of all eigen-oscillations with frequencies from the band $[\omega, \omega + \mathrm{d}\omega)$ has the value $\mathcal{P}^2(\omega, T)\mathrm{d}N_G(\omega)$, and, consequently, the average energy of the electromagnetic field in the domain G at temperature T can be computed as

$$\widetilde{E}_G(T) = 2\int \mathcal{P}^2(\omega, T)\mathrm{d}N_G(\omega), \tag{2.3}$$

where the factor "2" takes into account the existence of electromagnetic waves with two different polarizations. To make the last formula useful it is necessary to compute $\mathcal{P}^2(\omega, T)$ and $N_G(\omega)$.

In general, there is no way to obtain the exact value of $N_G(\omega)$ for an arbitrary domain G. However, if G has a sufficiently large volume V_G it can be estimated by the Weyl asymptote

$$N_G(\omega) \approx \frac{\omega^3 V_G}{6\pi^2 c^3} = \frac{4\pi V_G}{3\lambda^3}, \qquad \lambda = \frac{2\pi c}{\omega}, \tag{2.4}$$

which was conjectured in [7, 8], proved in [9, 10], and widely discussed in a vast literature including, but not limited to, a classical monograph [6, Ch.VI,§4] and a contemporary survey [11]. In the cases when the asymptote (2.4) is valid, the spectral frequencies ω_m are located close to each other and the difference $\mathrm{d}N_G(\omega)$ from (2.2) can be approximated as

$$\mathrm{d}N_G(\omega) = V_G\, D(\omega)\mathrm{d}\omega, \tag{2.5}$$

where

$$D(\omega) = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\omega^3}{6\pi^2 c^3}\right) = \frac{\omega^2}{2\pi^2 c^3} \tag{2.6}$$

is the quantity that is often referred to in the literature as the density of states [4].

From the above we see that the computation of $\mathrm{d}N_G(\omega)$ is a routine, although not trivial, mathematical problem that does not require external physical arguments. On the contrary, the average energy $\mathcal{P}^2(\omega, T)$ of the eigen-oscillator at frequency ω is computed based on the physically motivated quantum hypothesis that every eigen-oscillator has to be treated as a quantum oscillator, which, in particular, means that its energy $W(\omega)$ may take only one of the values

$$W(\omega) = W_n(\omega) = \hbar\omega\left(n + \frac{1}{2}\right), \qquad n = 0, 1, \cdots \tag{2.7}$$

where ω is the angular frequency of the oscillator and n is a non-negative integer.

After the quantum hypothesis is adopted the average energy $\mathcal{P}^2(\omega, T)$ of the eigen-oscillators at frequency ω and temperature T can be computed by standard methods of statistical physics. Thus, applying the Maxwell-Boltzmann-Gibbs statistics [4, 5] we find that any oscillator from the equilibrium ensemble at temperature T has the energy W_n with the probability

$$p_n = \frac{\mathrm{e}^{-W_n/\kappa T}}{\sum_{n=0}^{\infty} \mathrm{e}^{-W_n/\kappa T}}, \tag{2.8}$$

where κ is the Boltzmann constant, and the denominator is usually referred to as the statistical sum [3]. Then computing the average of the values W_n by the formula

$$\mathcal{P}^2(\omega, T) = \sum_{n=0}^{\infty} p_n W_n = \frac{\sum_{n=0}^{\infty} W_n \mathrm{e}^{-W_n/\kappa T}}{\sum_{n=0}^{\infty} \mathrm{e}^{-W_n/\kappa T}}, \tag{2.9}$$

we find that if an oscillator at frequency ω belongs to an equilibrium ensemble at temperature T, then its average energy has the value

$$\mathcal{P}^2(\omega, T) = \frac{\hbar\omega}{2} + P^2(\omega, T), \tag{2.10}$$

where the first term $\hbar\omega/2$ represents the so-called zero-point energy, which is not related to thermal processes, and the second term

$$P^2(\omega, T) = \frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/\kappa T} - 1} \tag{2.11}$$

represents the average thermal energy of the oscillator. Finally, substituting (2.10), (2.11) and (2.5) into (2.3) and replacing summation by integration, which is justified when the domain G is sufficiently large, we find that the average electromagnetic energy $\widetilde{E}_G(T)$ of the thermodynamically balanced domain G is defined by the integral

$$\widetilde{E}_G(T) = V_G \int \mathcal{E}(\omega, T)\mathrm{d}\omega \tag{2.12}$$

where

$$\mathcal{E}(\omega, T) = 2D(\omega)\mathcal{P}(\omega, T) \equiv \frac{\hbar\omega^3}{\pi^2 c^3}\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\hbar\omega/\kappa T} - 1}\right) \tag{2.13}$$

is the spectrum of the energy density (energy per unit volume per unit frequency band) of the equilibrium ensemble of electromagnetic oscillations at temperature T. It is obvious from (2.10) and (2.13) that the spectrum of energy density can be split into two components

$$\mathcal{E}(\omega, T) = \mathcal{E}_0(\omega) + \mathcal{E}_t(\omega, T), \tag{2.14}$$

where

$$\mathcal{E}_0(\omega, T) = \hbar\omega D(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3} \tag{2.15}$$

is the spectrum of the zero-point energy and

$$\mathcal{E}_t(\omega, T) = 2P^2(\omega, T)D(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3(\mathrm{e}^{\hbar\omega/\kappa T} - 1)} \tag{2.16}$$

is the spectrum of the thermal energy discovered by Planck.

3 Applicability of Planck's law and its meaning

Formula (2.13) coincides with one of the forms of Planck's celebrated law of black body radiation [1], which is widely regarded as one of the cornerstones of modern physics and technology. However, as shown above this is not a fundamental Law of Nature but a mathematical consequence derived from certain physical and mathematical assumptions and which, therefore, may not be valid unless all of the assumptions are met.

In particular, it is obvious that Planck's formula (2.13) is valid only if the quantity $\mathrm{d}N_G(\omega)$ from (2.3) can be represented as (2.5) where the density of states $D(\omega)$ is defined by (2.6). However, the derivation of the expression (2.6) is based on the Weyl asymptote (2.4) that is valid when the domain G is large compared to the dominant wavelength of the ensemble of oscillators. To estimate the importance of this restriction, we observe that the typical wavelength of thermal radiation at room temperature is about 600-1000nm, which means that Planck's law (1.1) can not be expected to hold in thermally isolated domains smaller than a few microns. For example, the electromagnetic field inside an insulated and thermally balanced cavity smaller than a few microns may not be expected to have the spectrum (1.1) but can be represented by the formula (2.13) with the density of states $D(\omega)$ computed by a more complex expression than the asymptote (2.6).

Another important limitation of Planck's formula is set in the Maxwell-Boltzmann-Gibbs statistics, which made it possible to get the key expressions (2.9) from the information about the energy levels (2.7) of the harmonic oscillator. Indeed, this statistics is strictly limited to systems in thermodynamical equilibrium, which implies that Planck's law is also limited to equilibrium systems. Therefore, it is not correct to apply Planck's law to the analysis of systems with a non-vanishing heat flux, and, as a result, this law can not be applied to the analysis of thermal conductance/resistance because to study thermal conductance it is necessary to deal with a system with a heat flux.

It is worth mentioning that although the statement that Planck's law describes the radiation from a domain in thermal equilibrium at temperature T is correct, it is nevertheless misleading because it does not emphasize that this law describes radiation from a domain only when this domain absorbs exactly the same amount of energy as it radiates [12, Lecture 42]. This peculiarity of Planck's law becomes especially visual in the example of two half-spaces A and B maintained at temperatures T_A and T_B, respectively. Thus, in the equilibrium case we not only know that $T_A = T_B$ but we can also use Planck's law to compute each of the mutually compensating energy fluxes $Q_{A\to B}$ and $Q_{B\to A}$ going from A to B and from B to A, respectively. On the contrary, if $T_A \neq T_B$ then Planck's law can not be used directly and we do not know $Q_{A\to B}$ and $Q_{B\to A}$ nor do we have any beforehand information about the net flux $Q = Q_{A\to B} - Q_{B\to A}$.

To illustrate further that Planck's law of thermal radiation can not be generally applied to the analysis of heat transport it suffices to consider the radiative heat transport through a vacuum gap of width H between two half-spaces $A = \{x, y, z :\ x < 0\}$ and $B = \{x, y, z :\ x > H\}$ occupied by identical materials.

Assume that the half-spaces A and B are maintained at the temperatures T_A and T_B, and that Planck's law of thermal radiation (2.13) is valid (which is not the case here). Then the energy densities of thermal electromagnetic radiation from the domains A and B have the spectra

$$\mathcal{E}_A(\omega, T_A) = \frac{2\hbar\omega D(\omega)}{\mathrm{e}^{\hbar\omega/\kappa T_A} - 1}, \qquad \mathcal{E}_B(\omega, T_B) = \frac{2\hbar\omega D(\omega)}{\mathrm{e}^{\hbar\omega/\kappa T_B} - 1}, \qquad \left(D(\omega) = \frac{\omega^2}{2\pi^2 c^3}\right), \tag{3.1}$$

respectively, and these radiations are spherically symmetrical in the sense that the radiation in any direction does not depend on the direction. Therefore, the energy flux between these half-spaces has the value

$$Q = 2\pi \int_0^{\pi/2} \left(\int_0^\infty \left[\mathcal{E}_A(\omega, T_A) - \mathcal{E}_B(\omega, T_B) \right] \mathrm{d}\omega \right) c \cos\theta \sin\theta \mathrm{d}\theta, \tag{3.2}$$

where θ is the angle between the direction of radiation and the x-axis, so that $v_x = c\cos\theta$ is the projection of the speed of light onto this axis. In the equilibrium case $T_A = T_B$, and so the net flux (3.2) through the gap vanishes. However, if $T_A \neq T_B$ then the net flow defined by (3.2) has a finite value that does not depend on the width of the gap and therefore remains constant even when the gap reduces to zero and becomes an imaginary plane in a single unbounded domain. This leads to the conclusion that an imaginary interface has a finite thermal resistance, which is obviously incorrect because otherwise a layer with an arbitrary thermal resistance could be constructed just by making a stack of imaginary gaps.

Since the above reasoning leads to a contradiction, we must conclude that it has a flaw, and this encourages us to find out where the flaw occurred. One obvious flaw is already discussed above — this is the unjustified application of Planck's law to a non-equilibrium system. There is also a popular opinion that the expression (3.2) leads to wrong conclusions because it does not take into account so called evanescent waves, which may propagate along one of the sides of the interface, and which are not accounted for in the spectra of thermal radiation (3.1). These two explanations of the failure of the expression (3.2) do not contradict each other, and so they can be explored together. However, simple reasonings presented below show that evanescent waves do not and can not carry heat across a gap, which essentially implies that the main thrust for understanding radiative heat transport is related with the extension of Planck's law to non-equilibrium systems. In order to prove this we need to define the field and describe the structure of evanescent waves in mathematical terms.

Let c be the phase speed of electromagnetic waves in the half-spaces $x < 0$ and $x > H$, and let c_0 be the speed of electromagnetic waves in the vacuum gap $0 < x < H$. Then, a plane wave

$$\phi_A = \mathrm{e}^{\mathrm{i}(e_x x + e_y y + e_z z)\omega/c}, \qquad e_x = \cos\theta \tag{3.3}$$

propagating in the half space $x < 0$ generates the plane wave

$$\phi_B = T(\theta) \mathrm{e}^{\mathrm{i}(e_x x + e_y y + e_z z)\omega/c} \tag{3.4}$$

propagating in the half-space $x > H$ and a more complicated field

$$\phi_0 = K_+(\theta) \mathrm{e}^{\mathrm{i}(d_x x + d_y y + d_z z)\omega/c_0} + K_-(\theta) \mathrm{e}^{\mathrm{i}(d_x (H-x) + d_y y + d_z z)\omega/c_0}, \tag{3.5}$$

which propagates in the vacuum layer $0 < x < H$. The directions of propagation of the waves from (3.3), (3.4) and (3.5) are related by

$$d_y = \gamma e_y, \qquad d_z = \gamma e_z, \qquad d_x = \sqrt{1 - \gamma^2 (e_y^2 + e_z^2)} \equiv \sqrt{1 - \gamma^2 \sin^2\theta}, \tag{3.6}$$

where

$$\gamma = \frac{c_0}{c}, \tag{3.7}$$

and $T(\theta)$ and $K_\pm(\theta)$ are generally complex-valued coefficients that can be computed by well known algorithms [13, 14].

It is easy to see that $e_x = \cos\theta$ must be real-valued. Indeed, if $\mathrm{Im}(e_x) > 0$, then the field ϕ_A exponentially grows as $x \to -\infty$, and if $\mathrm{Im}(e_x) < 0$ then ϕ_B exponentially grows as $x \to +\infty$. Therefore, the only physically admissible option is $\mathrm{Im}(e_x) = 0$, which, however, does not mean

d_x is always real. Indeed, if $\gamma > 1$, which is usually the case because electromagnetic waves in a medium propagate slower than in vacuum, we see that if the incidence angle θ satisfies the inequality $\gamma|\sin\theta| > 1$ then waves in the layer $0 < x < H$ have the structure

$$\widetilde{\phi}_0 = K_+(\mathrm{e}_x)\mathrm{e}^{-|d_x|x+\mathrm{i}(d_y y+d_z z)\omega/c_0} + K_-(\mathrm{e}_x)\mathrm{e}^{|d_x|(H-x)+\mathrm{i}(d_y y+d_z z)\omega/c_0}, \tag{3.8}$$

with imaginary $d_x = \mathrm{i}\sqrt{\gamma^2\sin^2\theta - 1}$. Waves of this type are referred to as evanescent waves and they play an important role in many areas of electromagnetic theory, but (3.8) makes it clear that these waves do not propagate along the x-axis and, thus, do not carry energy along this axis.

The above example shows that an unbounded increase of heat transport across a narrow gap cannot be explained by evanescent waves because such waves do not carry any heat across a gap. In the next example evanescent waves are not excited at all.

Consider a thin thermo-isolated pipe with its axis running along the x-axis. Let the parts $x < 0$ and $x > H$ of this pipe be filled by the same materials at temperatures T_A and T_B, let the piece $0 < x < H$ be a vacuum, and let the diameter of this pipe be smaller than the wavelength of most heat carrying electromagnetic radiation waves. Then this pipe can be considered as a one-dimensional version of the system considered above, and, if we assume that Planck's radiation law is valid, then the radiative heat flux across the gap in the pipe is also described by the formula (3.2), but with the one-dimensional density of states $D(\omega) = 1/2c$. This leads to the same paradoxical predictions as appear in the three-dimensional case, but this time they can not be attributed to evanescent waves because they do not exist in one-dimensional systems.

The above considerations show that the increase of radiative heat transport across a gap with decreasing width can not be rationally explained in terms of the classical Planck's law, which is applicable only to equilibrium systems without any heat flux. Therefore, if we wish to have a reliable tool for the analysis of radiative heat transport, then we need to extend Planck's law to systems with a non-vanishing heat flux.

4 Ensembles of electromagnetic waves with a constant heat flux

The derivation of Planck's radiation law outlined in Section 2 does not use any specific information about the structure of eigen-oscillations and due to this generality the law remains valid in any sufficiently large domain. The entire space is definitely "sufficiently large", but it is also so simple that all of its eigen-oscillations can be described by an explicit formula

$$\Phi(\vec{r},t) = \iiint_{\mathbb{R}^3} F(\vec{k})\mathrm{e}^{2\pi\mathrm{i}\vec{k}\cdot\vec{r}-\mathrm{i}\omega t}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z, \tag{4.1}$$

where $\vec{r} = (x, y, z)$ is the vector-point of the observer, $\vec{k} = (k_x, k_y, k_z)$ is the wave-vector related to the frequency ω by the dispersion relation

$$|\vec{k}| = \frac{\omega}{2\pi c}, \tag{4.2}$$

and $F(\vec{k})$ is an arbitrary integrable function of $\vec{k}$. Not surprisingly, with the explicit description of the eigen-oscillations at hand, we can get more information about the thermal radiation in the entire space than is offered by Planck's formula (2.13).

If $F(\vec{k})$ is known then the integral (4.1) represents a field with the energy density

$$E = 2\iiint_{\mathbb{R}^3} |F(\vec{k})|^2\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z, \tag{4.3}$$

where the factor “2” is added to take into account the existence of electromagnetic fields with two different polarizations. If the amplitude $F(\vec{k})$ is random, then we can consider the statistical ensemble of the fields (4.1) and compute its average energy density as

$$E = 2\iiint_{\mathbb{R}^3} \langle |F(\vec{k})|^2\rangle \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}k_z = 2\int_{\mathbb{S}}\int_0^\infty \langle |F(\vec{k})|^2\rangle k^2 \mathrm{d}k\ \mathrm{d}S(\vec{e}), \tag{4.4}$$

where $\vec{e}$ is the unit vector parallel to the wave vector $\vec{k}$, and $\mathrm{d}S(\vec{e})$ is an area element on the unit sphere $\mathbb{S} = \{\vec{e}:\ |\vec{e}| = 1\}$.

Let the ensemble consisting of the fields (4.1) with random coefficients be in thermodynamical equilibrium at temperature T. Then, the average energy density of the electromagnetic field described by (4.1) can be computed from the quantum mechanical point of view. Since the plane wave $\mathrm{e}^{2\mathrm{i}\pi\vec{k}\cdot\vec{r}-\mathrm{i}\omega t}$ appears as a harmonic oscillator at frequency ω we adopt the quantum hypothesis and conclude that if this oscillator belongs to an equilibrium ensemble at temperature T then its average energy density has the value $\mathcal{P}^2(\omega, T)$ from (2.10). On the other hand, the average energy density of the ensemble (4.1) can also be represented by the integral

$$E = 2\int_0^\infty \mathcal{P}^2(\omega, T)\mathrm{d}V(\omega), \tag{4.5}$$

where $V(\omega)$ is the volume of the domain occupied by the wave vectors $\vec{k}$ corresponding to the frequencies below ω. It is obvious from (4.2) that

$$V(\omega) = \frac{4\pi}{3}\left(\frac{\omega}{2\pi c}\right)^3 \equiv \int_{\mathbb{S}}\left(\int_0^k \xi^2 \mathrm{d}\xi\right)\mathrm{d}S(\vec{e}), \qquad k = \frac{\omega}{2\pi c}. \tag{4.6}$$

Then, differentiating (4.6) by k we find that

$$\frac{\mathrm{d}V(\omega)}{\mathrm{d}k} = \int_{\mathbb{S}} k^2\,\mathrm{d}S(\vec{e}), \qquad \omega = 2\pi ck. \tag{4.7}$$

Straightforward substitution of (4.7) into (4.5) leads to the expression

$$E = 2\int_{\mathbb{S}}\int_0^\infty \mathcal{P}^2(2\pi ck, T)k^2\mathrm{d}k\,\mathrm{d}S(\vec{e}), \tag{4.8}$$

whose comparison with (4.4) results in the equation

$$\left\langle |F(\vec{k})|^2\right\rangle = \mathcal{P}^2(2\pi c|\vec{k}|, T), \tag{4.9}$$

which determines the average of the amplitudes of the ensemble of electromagnetic fields (4.1) in thermal equilibrium at temperature T.

If the ensemble of vibrations has a heat flux then it cannot be described by the formula (4.9) derived from the assumption of thermodynamical equilibrium. However, the method that led to this expression can be adjusted to handle non-equilibrium systems with a constant heat flux.

To analyze the ensemble of electromagnetic waves with a constant heat flux $\vec{Q}$ we first consider it in a moving frame. Let $\vec{v}$ be the constant velocity of the moving frame, which means that the observer's positions $\vec{r}$ and $\vec{r}_1$ in the reference and the moving frames are related by the formula

$$\vec{r}_1 = \vec{r} - \vec{v}t, \tag{4.10}$$

illustrated in Fig. 1. Then, applying a well-known method of computation of the energy characteristics of the electromagnetic field in a moving frame [15] we find that the heat flux $\vec{Q}_1$ in the moving frame has the value

$$\vec{Q}_1 = \vec{Q} - \vec{v}E_t + o(\vec{v}^2), \tag{4.11}$$

where $\vec{Q}$ is the heat flux in the reference frame and

$$E_t = \int_0^\infty \mathcal{E}_t(\omega, T)\mathrm{d}\omega, \tag{4.12}$$

is the density of thermal energy represented in terms of the spectrum of thermal energy $\mathcal{E}_t(\omega, T)$ from (2.16). If $\vec{v} = \vec{Q}/E_t$ then the ensemble of vibrations in the moving frame has no heat flux and can, therefore, be described by the methods of equilibrium thermodynamics.

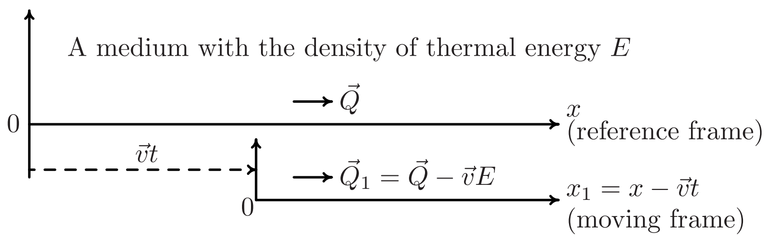

Figure 1: Heat flux in the reference and moving coordinate frames

Since a moving frame is involved in the analysis it is legitimate to raise a question about the consistent formulation of thermodynamics in the relativistic framework and, in particular, about the relationship between the temperatures T_v and T in the moving and the reference frames. This question attracted immediate attention after the formulation of the special theory of relativity, and in 1907 M.Planck [16] and Einstein [17] concluded that $T_v = T\sqrt{1 - v^2/c^2}$, where c is the speed of light and v is the speed of the moving frame. Much later, in 1963 this result was challenged [18] by the opposite suggestion $T = T_v\sqrt{1 - v^2/c^2}$, and in 2003 it was argued [19] that $T_0 = T$. It should be mentioned that the scope of the debates was much broader than the relativistic definition of the temperature and included discussions of the relativistic nature of other thermodynamical concepts such as entropy, heat flux, the thermal energy-momentum four-vector, just to mention a few [20, 21, 22]. Moreover, in [20, 21] the proposed "third" formulation of relativistic thermodynamics was applied to the analysis of thermal exchange between systems moving with different speeds.

The disagreement between the above mentioned theories was broadly discussed, and it was recently shown [23, 24, 25] that the ambiguities arise because continuum theories of matter rely on such notions as spatial density, which breaks the symmetry between time and space required by the theory of relativity. After identifying the cause of such ambiguity, the paper [24] proposed a new insightful approach that resolved the deficiencies of the previous theories and was testable by many-body computer simulations and, in principle, by physical experiments.

Despite the importance of the consistent formulation of relativistic thermodynamics and of the recent successes in this direction, our analysis is not directly affected by the relativistic definition of temperature. Indeed, in all of the existing theories, the difference $T_v - T$ between the temperature T in the reference frame and the temperature T_v in the frame moving with the speed v is on the order of v^2/c^2, which is negligible in our analysis where the speed $v \approx Q/E$ is proportional to the heat flux Q, which is assumed to be small compared to the speed of light. Therefore, for our purpose it suffices to assume that the temperature in the moving and reference frames are equal i.e., $T_v = T$.

Taking into account (4.10) we find that in terms of $\vec{r}_1$ the field (4.1) can be represented as

$$\Phi(\vec{r},t) = \widetilde{\Phi}(\vec{r}_1,t) = \iiint_{\mathbb{R}^3} F(\vec{k}) \mathrm{e}^{2\pi\mathrm{i}\vec{k}\cdot\vec{r}_1 - \mathrm{i}\omega_1 t} \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}k_z, \tag{4.13}$$

where the modified frequency ω_1 and the wave vector $\vec{k}$ are related by the dispersion equation

$$\omega_1 = \omega q(\vec{k}) \equiv 2\pi c|\vec{k}|q(\vec{k}), \tag{4.14}$$

which includes the Doppler factor

$$q(\vec{k}) = 1 - \frac{\vec{Q}\cdot\vec{k}}{cE_t}. \tag{4.15}$$

Since the ensemble (4.13) is in equilibrium its energy density has the Planck spectrum

$$E = 2\int_0^\infty \mathcal{P}^2(\omega_1,T)\mathrm{d}V_1(\omega_1), \tag{4.16}$$

where

$$V_1(\omega_1,Q) = \int_{\mathbb{S}} \left(\int_0^k \xi^2 \mathrm{d}\xi\right) \mathrm{d}S(\vec{e}), \qquad \omega_1 = 2\pi c q(k\vec{e})k, \tag{4.17}$$

is the volume of the domain $k \equiv |\vec{k}| < \omega_1/2\pi c q(\vec{k})$. Combining the last two formulas we find that

$$E = 2\int_{\mathbb{S}}\int_0^\infty \mathcal{P}^2(2\pi c q(k\vec{e})k, T)\, k^2 \mathrm{d}k\, \mathrm{d}S(\vec{e}), \tag{4.18}$$

and then comparing (4.18) with (4.4) we get the relationship

$$\langle |F(\vec{k})|^2 \rangle = \mathcal{P}^2\left(\omega\left(1 - \frac{\vec{Q}\cdot\vec{k}}{E_t}\right), T\right) \qquad \omega = 2\pi c|\vec{k}|, \tag{4.19}$$

which generalizes (4.9) and determines the average amplitude of the ensemble of electromagnetic fields (4.1) with the constant heat flux $\vec{Q}$ at temperature T.

It is easy to see that (4.19) provides much more information about electromagnetic radiation than Planck's formula (2.13). Indeed, while (2.13) describes the spectrum of the total amount of thermal radiation in all directions, our result (4.18), (4.19) makes it possible to compute the spectrum of thermal radiation in each direction, which are not equal in the presence of a heat flux. To see this we substitute (4.18) into (4.4) and represent the result in the form

$$E = \int_{\mathbb{S}}\int_0^\infty \mathcal{E}(\omega,T,\vec{e},\vec{Q})\, \mathrm{d}S(\vec{e})\, \mathrm{d}\omega \tag{4.20}$$

where

$$\mathcal{E}(\omega,T,\vec{e},\vec{Q}) = \frac{\omega^2}{4\pi^3 c^3}\mathcal{P}^2\left(\omega Y(\vec{e},\vec{Q}), T\right) \tag{4.21}$$

with the coefficient

$$Y(\vec{e},\vec{Q}) = 1 - \frac{\vec{Q}\cdot\vec{e}}{cE_t}, \tag{4.22}$$

has the meaning of the spectrum of radiation along the direction $\vec{e}$ in the presence of the heat flux $\vec{Q}$ (energy per unit volume, per unit frequency band, per unit of solid angle). To continue the analogy with Planck's radiation we use (2.10) and split the directional spectrum (4.21) as

$$\mathcal{E}(\omega,T,\vec{e},\vec{Q}) = \mathcal{E}_0(\omega,\vec{e},\vec{Q}) + \mathcal{E}_t(\omega,T,\vec{e},\vec{Q}) \tag{4.23}$$

where

$$\mathcal{E}_0(\omega, \vec{e}, \vec{Q}) = \frac{\hbar\omega^3\, Y(\vec{e}, \vec{Q})}{8\pi^3 c^3}, \tag{4.24}$$

is the directional spectrum of the zero-point energy, and

$$\mathcal{E}_t(\omega, T, \vec{e}, \vec{Q}) = \frac{\hbar\omega^3\, Y(\vec{e}, \vec{Q})}{4\pi^3 c^3 \left(\mathrm{e}^{\hbar\omega Y(\vec{e},\vec{Q})/\kappa T} - 1\right)} \tag{4.25}$$

is the directional spectrum of thermal energy, which extends Planck's law (1.1) from equilibrium to steady-state systems with a constant heat flux Q at constant temperature T.

5 Discussion

The phenomena of heat transport by means of propagating waves, such as electromagnetic radiation or mechanical elastic waves, were known and studied for decades, and, with one noticeable exception, i.e., the Kapitsa thermal resistance between highly dissimilar materials, theoretical predictions of thermal transport were rather close to experimental data. However, since the beginning of exploration of the micro and nano scale worlds, disagreements between the theory and measurements of heat transport have become more common and severe, and it became important to understand why the theories successfully used in the past failed to explain new experiments. For example, the theory based on the classical Planck's law accurately predicts radiative heat transport between two half-spaces separated by a sufficiently wide gap, but if applied to a narrow gap, this theory gives errors that may reach up to two orders of magnitude. Similarly, conventional theories give rather good predictions of heat transport by acoustic waves (phonons) through wide cross-section silicon wires but give significant errors when the wire becomes narrower than a few tens of nanometers.

To understand the peculiarities of radiative heat transport in nano and micro scale structures it suffices to recall that thermally excited waves have bounded wavelengths and that fields composed from such waves admit decomposition into wave-packets [26, 27] characterized by the dominant wavelength and with its size considerably larger than this wavelength. It is well known [26] that if the characteristic dimensions of the medium are considerably larger than the wave packets, which are in turn considerably larger than their dominant wavelength, then such packets can be treated as particles that transport energy in accordance with the kinetic theory of gases. This condition is satisfied in the case of thermal radiation across a wide gap, but it is not satisfied when the gap becomes sufficiently narrow.

It is important to note that in order to be treated by the kinetic theory of gases wave packets must be small not only compared to the size of the domain, but they must also be small enough to occupy a small portion of the space, which can not be guaranteed even in considerably large structures. One such situation is illustrated in Fig. 2 which shows the evolution of wave-packets at the interface between materials with very different wave speeds. In this figure the incident ball-shaped wave packet "I" arrives at the interface from the left side, which is occupied by a medium with a low sound speed. After interaction with the interface this packet splits into the reflected and transmitted wave-packets "R" and "T", which propagate in the slow and the fast media, respectively. The size of the reflected packet "R" remains similar to that of the incident packet, but the transmitted packet "T" becomes larger than the incident packet, and, if the contrast between the materials is sufficiently large then this packet may become too large to be treated as a gas particle. In this example, the kinetic theory can not be applied even though the structure does not have small characteristic dimensions.

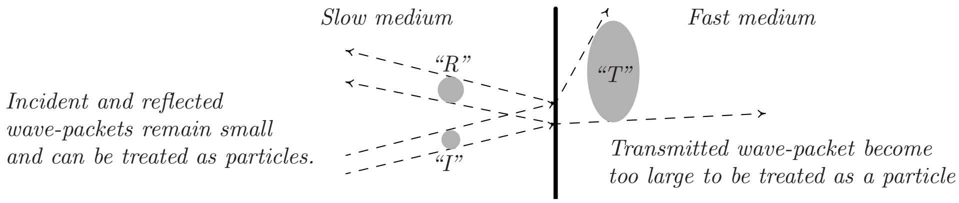

Figure 2: Destruction of wave-packets at an interface.

In the cases when the kinetic theory of gases can be employed, the process of radiative heat transport is not affected by the wave properties of radiation. In these cases, the frequency of the radiation is just a quantity characterizing the energy of a wave-packet, and the interference between packets is negligible because it happens only during rare collisions that take a small fraction of the packet's lifetime. However, if thermally excited waves have wavelengths so long that the corresponding wave packets are not small enough to be treated as particles, then the wave packets interfere with each other as well as with their own reflections from boundaries and interfaces. In these cases the wave properties of thermal radiation can not be ignored, which, in particular, means, that the spectrum of thermal radiation in the presence of a heat flux must be represented by the extended Planck's law, which differs from the classical equilibrium Planck's law by such typical attributes of waves as the Doppler transformation.

It follows from the above that the extension (4.25) of Planck's law has to be applied for the analysis of radiative heat transport in all structures where wave phenomena are important. This includes problems of radiative heat transport across narrow gaps, across multi-layered structures with thin layers, across interfaces between sharply dissimilar media, in narrow wires and pipes, and many other nano and micro scale structures.

It should be emphasized that the extension of Planck's law presented here is not limited to the radiation of electromagnetic waves but can be applied for the analysis of thermal transport by any type of waves, including acoustical waves, which are known to be primary heat carriers between dielectric materials in direct contact. Moreover, a similar extension can be derived for the statistical distribution of the electron emission in the presence of a non-vanishing current, which is important for the analysis of electron transport in micro and nano scale devices.

We have applied the proposed extension of Planck's law to the analysis of Kapitsa thermal resistance and obtained results [28, 29] that show correct order of magnitude Kapitsa resistance for the first time without any added elements such as surface roughness or nonlinearities, and which is almost two orders of magnitude more accurate than conventional theories based on the classical Planck's law. In a closely related paper [30] we considered radiative heat transport between two dielectric half-spaces separated by a gap of small width d, and we obtained agreement with recent experimental observation, that the conductance becomes unbounded as $O(1/d^2)$ between two materials at different temperatures as the gap's width d vanishes. This was done without employing extraneous point sources, roughness, or non-linearities, which are often introduced to explain the difference between experimental data and the conventional theories. A planned forthcoming paper will demonstrate that the observed asymmetry of heat conductance through an interface which is associated in [31, 32] with specific non-linear effects, appears as a natural consequence of the modified Planck's law, which however, does not exclude the influence of other factors studied in[31, 32]. Another forthcoming paper will demonstrate that the intriguing behavior of thermal conductivity of silicon nanowires at low temperatures reported in [33], can also be rationally explained by the modification of Planck's law discussed above.

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