A μ-synthesis approach to guaranteed cost control in track-following servos

Richard Conway and Roberto Horowitz

Computer Mechanics Laboratory

Mechanical Engineering

University of California, Berkeley, CA, USA

January 21, 2009

Abstract

This paper presents a new control synthesis approach for dual-stage track-following servo systems with multi-rate sensing and actuation. For these systems, the robust track-following problem can be formulated as a periodic time-varying guaranteed cost control problem. To reduce the conservatism of the guaranteed cost control framework, uncertainty scalings such as those used in the D-K iteration heuristic for μ-synthesis are introduced. Although this results in a non-convex optimization problem, it is shown that it lends itself to a methodology similar to D-K iteration. Using this methodology, a controller is designed for a set of hard disk drives which minimizes the worst-case ℓ_2 semi-norm performance of the system.

Introduction

For several decades now, the areal storage density of hard disk drives (HDDs) has been doubling roughly every 18 months, as predicted by Kryder's law. As the storage density is pushed higher, the concentric tracks on the disk which contain data must be pushed closer together, which requires much more accurate control of the read/write head. The current areal storage density of hard drives, as reported by [4], is 345 gigabits/in^2.

The current goal of the magnetic recording industry is to achieve an areal storage density of 1 terabit/in^2. It is expected that the track width required to achieve this data density is $46nm$. This means that the 3σ value of the closed loop position error signal (PES) should be less than $4.6nm$ to achieve this specification. To help achieve this goal, the use of a secondary actuator has been proposed to give increased precision in read/write head positioning. In this paper, as in [7], we use a microactuator (MA) which directly actuates the head/slider assembly with respect to the suspension tip and generates measurements of the MA displacement (RPES). This allows for the design of higher bandwidth controllers which give smaller PES magnitudes.

However, since there tend to be large variations in HDD dynamics due to variations in manufacture and assembly, it is not enough to achieve this performance for a single plant; the controller must guarantee the desired level of performance for a large set of HDDs. Thus, we are interested in finding a controller which gives robust performance over a set of HDDs. One framework for solving this problem is guaranteed cost control. This methodology is a multiobjective control design methodology whose objectives involve worst-case quadratic costs over a modeled set of parametric uncertainty. Both the state feedback synthesis problem and the output feedback synthesis problem can be solved using convex optimization involving linear matrix inequalities (LMIs), as in [9] and [3], respectively. Like $\mathcal{H}_\infty$ control, however, guaranteed cost control is unable to take advantage of the uncertainty structure in the model. This motivates the usage of uncertainty scalings, as are used in the D-K iteration heuristic for μ-synthesis.

This paper derives the relevant conditions for guaranteed cost control of HDDs with uncertainty scalings and shows that this results in a non-convex controller optimization problem. However, the problem becomes convex when the values of either of two sets of variables is fixed. Trivially, the problem becomes convex when the uncertainty scalings are fixed. However, unlike D-K iteration, not all of the control parameters need to be fixed in order to optimize the uncertainty scalings. Based on this, a control design methodology similar to D-K iteration is presented and applied to the design of a robust HDD controller.

Guaranteed Cost Control

Preliminaries

Throughout this paper, we will be considering discrete time linear periodically time-varying (LPTV) systems which admit a periodic time-varying state space realization.We will denote the state space realization of an LPTV system, H^d, by

$$H^d \sim \left(\begin{array}{c|c} A_k^d & B_k^d \\ \hline C_k^d & D_k^d \end{array}\right)$$

where the subscript refers to the time index and it is assumed that all state space matrices are periodic with period N, e.g. $A_{N+1}^d = A_1^d$.

For linear time-invariant (LTI) systems, a commonly used measure of performance is the $\mathcal{H}_2$ norm. In the time domain, if z^{LTI} is the output of an LTI system H^{LTI}, the $\mathcal{H}_2$ norm has the interpretation of being the root-sum-square standard deviation of the elements of z^{LTI} when H^{LTI} is driven by zero-mean white Gaussian noise with unit covariance. However, for an LPTV system H^d with output z^d, the second-order statistics of z^d vary periodically with time. Thus, what we are interested in is the root-mean-square value (over time) of the root-sum-square standard deviation of the elements of z^d. This corresponds to the definition of the ℓ_2 semi-norm given by

$$\left\|H^d\right\|_2^2 := \limsup_{l\to\infty} \frac{1}{2l+1} \sum_{k=-l}^{l} \operatorname{tr} \mathcal{E}\left[z_k^d \left(z_k^d\right)^T\right].$$

Thus, we are interested in finding an upper bound on the ℓ_2 semi-norm of a LPTV system. The following lemma, which is proved, for example, in [2], gives a sharp upper bound on the ℓ_2 semi-norm of a LPTV system.

Lemma 1. $\|H^d\|_2^2 < \gamma \Leftrightarrow \exists P_k, W_k$ *such that*

$$\gamma > \frac{1}{N}\sum_{k=1}^{N} \operatorname{tr} W_k$$

$$\begin{bmatrix} W_k & C_k^d P_k & D_k^d \\ \bullet & P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0$$

$$\begin{bmatrix} P_{k+1} & A_k^d P_k & B_k^d \\ \bullet & P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0$$

for $k = 1, \ldots, N$ *where bullets represent elements which follow from symmetry and* $P_{N+1} = P_1$.

With some manipulation, these matrix inequalities can be shown to be equivalent to those obtained using a lifting procedure such as the one used by [6]. The big difference between these two methodologies is that the matrices in the lifting approach are large and sparse whereas the ones here are dense and have low dimension, i.e. they exploit the relevant sparsity structure. Thus, working with periodic systems will result in more efficient optimization schemes.

Although this paper does not explicitly consider multiobjective control design, one of the goals is to present a methodology which will trivially extend to that case. Thus, we would like to minimize the conservatism of our analysis when performing multiobjective control design. For LTI systems, it was shown in [8] that the use of an extended norm characterization using an instrumental variable reduces the conservatism introduced in multiobjective controller design. Thus, in this paper we will be using the following lemma which can be deduced from the previous one using the methodology in [8].

Lemma 2. $\|H^d\|_2^2 < \gamma \Leftrightarrow \exists P_k, G_k, W_k$ *such that*

$$\gamma > \frac{1}{N}\sum_{k=1}^{N} \operatorname{tr} W_k \tag{1a}$$

$$\begin{bmatrix} W_k & C_k^d G_k & D_k^d \\ \bullet & G_k + G_k^T - P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0 \tag{1b}$$

$$\begin{bmatrix} P_{k+1} & A_k^d G_k & B_k^d \\ \bullet & G_k + G_k^T - P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0 \tag{1c}$$

for $k = 1, \ldots, N$ *where* $P_{N+1} = P_1$.

Analysis LMIs

In this section, we develop the necessary theory to derive an upper bound on the worst-case ℓ_2 semi-norm over all modeled uncertainty. First we define the LPTV system

$$
\begin{gathered}
H^{\Delta} \sim \left(\begin{array}{c|c} A_k^{\Delta} & B_k^{\Delta} \\ \hline C_k^{\Delta} & D_k^{\Delta} \end{array}\right) := \left(\begin{array}{c|c} \overline{A}_k + \Phi_k^{AB}\overline{E}_k^{11} & \overline{B}_k + \Phi_k^{AB}\overline{E}_k^{12} \\ \hline \overline{C}_k + \Phi_k^{CD}\overline{E}_k^{21} & \overline{D}_k + \Phi_k^{CD}\overline{E}_k^{22} \end{array}\right) \\
\Phi_k^{AB} := \overline{E}_k^{13}\Delta_k^{AB}\left(I - \overline{E}_k^{14}\Delta_k^{AB}\right)^{-1} \\
\Phi_k^{CD} := \overline{E}_k^{23}\Delta_k^{CD}\left(I - \overline{E}_k^{24}\Delta_k^{CD}\right)^{-1}
\end{gathered}
\tag{2}
$$

where $\overline{A}_k, \overline{B}_k, \overline{C}_k, \overline{D}_k, \overline{E}_k^{ij}$ are known matrices with period N, and the matrices Δ_k^{AB} and Δ_k^{CD} are unknown real matrices (not necessarily periodic) with the form

$$
\begin{aligned}
\Delta_k^{AB} \in \underline{\Delta}^{AB} &:= \left\{\operatorname{diag}\left[\alpha^1 I_{m(1)}, \ldots, \alpha^q I_{m(q)}\right] : \left|\alpha^i\right| \le 1\right\} \\
\Delta_k^{CD} \in \underline{\Delta}^{CD} &:= \left\{\operatorname{diag}\left[\beta^1 I_{n(1)}, \ldots, \beta^r I_{n(r)}\right] : \left|\beta^i\right| \le 1\right\}.
\end{aligned}
$$

This form could be arrived at, for example, by expressing $[A_k^{\Delta} B_k^{\Delta}]$ as an LFT, expressing $[C_k^{\Delta} D_k^{\Delta}]$ as an LFT, and then bringing the parametric uncertainty in both LFTs inside of their respective loops.

We now define the sets of matrices

$$
\begin{aligned}
\underline{S}^{AB} &:= \left\{ \operatorname{diag}[D^1, \ldots D^q] : \begin{array}{c} D^i \in \mathbb{R}^{m(i)\times m(i)} \\ \det D^i \neq 0 \end{array} \right\} \\
\underline{S}^{CD} &:= \left\{ \operatorname{diag}[D^1, \ldots D^r] : \begin{array}{c} D^i \in \mathbb{R}^{n(i)\times n(i)} \\ \det D^i \neq 0 \end{array} \right\}
\end{aligned}
$$

and note that for each $\Delta_k^{AB} \in \underline{\Delta}^{AB}$ and $\Delta_k^{CD} \in \underline{\Delta}^{CD}$

$$
\begin{aligned}
\Delta_k^{AB} &= S^{AB}\Delta_k^{AB}\left(S^{AB}\right)^{-1}, \quad \forall S^{AB} \in \underline{S}^{AB} \\
\Delta_k^{CD} &= S^{CD}\Delta_k^{CD}\left(S^{CD}\right)^{-1}, \quad \forall S^{CD} \in \underline{S}^{CD}
\end{aligned}
$$

i.e. any matrix in these sets will commute with their respective uncertainty matrices.

Also, for each $S^{AB} \in \underline{S}^{AB}$ and $S^{CD} \in \underline{S}^{CD}$, we define the sets of unstructured uncertainty

$$
\begin{aligned}
\underline{\Delta}^{AB,u}\left(S^{AB}\right) &:= \left\{\Delta : \left\|\left(S^{AB}\right)^{-1}\Delta S^{AB}\right\| \le 1\right\} \\
\underline{\Delta}^{CD,u}\left(S^{CD}\right) &:= \left\{\Delta : \left\|\left(S^{CD}\right)^{-1}\Delta S^{CD}\right\| \le 1\right\}
\end{aligned}
$$

where $\|\Delta\|$ denotes the maximum singular value of Δ. Note that

$$\begin{aligned} \underline{\Delta}^{AB} &\subset \underline{\Delta}^{AB,u}\left(S^{AB}\right), \quad \forall S^{AB} \in \underline{S}^{AB} \\ \underline{\Delta}^{CD} &\subset \underline{\Delta}^{CD,u}\left(S^{CD}\right), \quad \forall S^{CD} \in \underline{S}^{CD}. \end{aligned} \tag{3}$$

With this notation in place, we can now state and prove two lemmas which we will need to find an upper bound on the guaranteed ℓ_2 semi-norm performance of an uncertain LPTV system.

Lemma 3. *If $\overline{E}_k^{23} \neq 0$ and $S_k \in \underline{S}^{CD}$, then the following conditions are equivalent:*

1. *$\exists P_k, G_k, W_k$ such that*

$$\begin{bmatrix} W_k & C_k^{\Delta} G_k & D_k^{\Delta} \\ \bullet & G_k + G_k^T - P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0, \quad \forall \Delta_k^{CD} \in \underline{\Delta}^{CD,u}\left(S_k\right) \tag{4}$$

2. *$\exists P_k, G_k, W_k, \tau$ such that*

$$\begin{bmatrix} \tau S_k S_k^T & 0 & \overline{E}_k^{21} G_k & \overline{E}_k^{22} & \tau \overline{E}_k^{24} S_k S_k^T \\ \bullet & W_k & \overline{C}_k G_k & \overline{D}_k & \tau \overline{E}_k^{23} S_k S_k^T \\ \bullet & \bullet & G_k + G_k^T - P_k & 0 & 0 \\ \bullet & \bullet & \bullet & I & 0 \\ \bullet & \bullet & \bullet & \bullet & \tau S_k S_k^T \end{bmatrix} \succ 0 \tag{5}$$

Proof. First we define for convenience

$$\Psi := G_k \left(G_k + G_k^T - P_k\right)^{-1} G_k^T$$

$$L := \begin{bmatrix} 0 & 0 \\ 0 & W_k \end{bmatrix} - \begin{bmatrix} \overline{E}_k^{21} & \overline{E}_k^{22} \\ \overline{C}_k & \overline{D}_k \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{E}_k^{21} & \overline{E}_k^{22} \\ \overline{C}_k & \overline{D}_k \end{bmatrix}^T.$$

Via Schur complements, it is easily verified that (4) holds if and only if $\Psi \succ 0$ and

$$\begin{bmatrix} \Phi_k^{CD} & I \end{bmatrix} L \begin{bmatrix} \Phi_k^{CD} & I \end{bmatrix}^T \succ 0, \quad \forall \Delta_k^{CD} \in \underline{\Delta}^{CD,u}\left(S_k\right).$$

Defining $\xi := (\Phi_k^{CD})^T x$, the previous condition holds if and only if $\Psi \succ 0$ and

$$\begin{bmatrix} \xi \\ x \end{bmatrix}^T L \begin{bmatrix} \xi \\ x \end{bmatrix} > 0, \quad \forall x \neq 0, \Delta_k^{CD} \in \underline{\Delta}^{CD,u}(S_k). \tag{6}$$

Now note that

$$\xi = \left[I - (\Delta_k^{CD})^T \left(\overline{E}_k^{24}\right)^T \right]^{-1} (\Delta_k^{CD})^T \left(\overline{E}_k^{23}\right)^T x$$

$$\Rightarrow \xi = \left(\Delta_k^{CD}\right)^T \begin{bmatrix} \overline{E}_k^{24} \\ \overline{E}_k^{23} \end{bmatrix}^T \begin{bmatrix} \xi \\ x \end{bmatrix}$$

$$\Rightarrow S_k^T \xi = \left(S_k^{-1} \Delta_k^{CD} S_k\right)^T S_k^T \begin{bmatrix} \overline{E}_k^{24} \\ \overline{E}_k^{23} \end{bmatrix}^T \begin{bmatrix} \xi \\ x \end{bmatrix}.$$

Thus, $\Delta_k^{CD} \in \underline{\Delta}^{CD,u}(S_k)$ if and only if

$$\xi^T S_k S_k^T \xi \leq \begin{bmatrix} \xi \\ x \end{bmatrix}^T \begin{bmatrix} \overline{E}_k^{24} \\ \overline{E}_k^{23} \end{bmatrix} S_k S_k^T \begin{bmatrix} \overline{E}_k^{24} \\ \overline{E}_k^{23} \end{bmatrix}^T \begin{bmatrix} \xi \\ x \end{bmatrix}.$$

Since $\overline{E}_k^{23} \neq 0$, $\exists \xi, x$ such that the above inequality is strict. Thus, we can use the $\mathcal{S}$-procedure (see [1]) to say that (6) holds if and only if $\exists \tau > 0$ such that

$$\begin{bmatrix} \xi \\ x \end{bmatrix}^T L \begin{bmatrix} \xi \\ x \end{bmatrix} > \tau \begin{bmatrix} \xi \\ x \end{bmatrix}^T \left(\begin{bmatrix} \overline{E}_k^{24} \\ \overline{E}_k^{23} \end{bmatrix} S_k S_k^T \begin{bmatrix} \overline{E}_k^{24} \\ \overline{E}_k^{23} \end{bmatrix}^T - \begin{bmatrix} S_k S_k^T & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \xi \\ x \end{bmatrix}, \quad \forall \xi, x \neq 0.$$

It is straightforward to show using Schur complements that this condition along with the condition $\Psi \succ 0$ is equivalent to (5), which concludes the proof. □

Lemma 4. *If $\overline{E}_k^{13} \neq 0$ and $S_k \in \underline{S}^{AB}$, then the following conditions are equivalent:*

1. *$\exists P_k, G_k$ such that*

$$\begin{bmatrix} P_{k+1} & A_k^{\Delta} G_k & B_k^{\Delta} \\ \bullet & G_k + G_k^T - P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0, \quad \forall \Delta_k^{AB} \in \underline{\Delta}^{AB,u}(S_k) \tag{7}$$

2. $\exists P_k, G_k, \tau$ *such that*

$$\begin{bmatrix} \tau S_k S_k^T & 0 & \overline{E}_k^{11} G_k & \overline{E}_k^{12} & \tau \overline{E}_k^{14} S_k S_k^T \\ \bullet & P_{k+1} & \overline{A}_k G_k & \overline{B}_k & \tau \overline{E}_k^{13} S_k S_k^T \\ \bullet & \bullet & G_k + G_k^T - P_k & 0 & 0 \\ \bullet & \bullet & \bullet & I & 0 \\ \bullet & \bullet & \bullet & \bullet & \tau S_k S_k^T \end{bmatrix} \succ 0$$

The proof of this lemma is omitted because it is nearly identical to the proof of the previous lemma.

We now take a minute to discuss the relevance of the technical condition in Lemma 3 that $\overline{E}_k^{23}$ must be nonzero. Suppose that $\overline{E}_k^{23} = 0$. Then $\Phi_k^{CD} = 0$, which in turn means that $C_k^\Delta = \overline{C}_k$ and $D_k^\Delta = \overline{D}_k$. In other words, it is trivial to directly check whether or not (4) holds because C_k^Δ and D_k^Δ are known. Thus, even though the lemma does not apply in this case, there is still a way to account for it without introducing conservatism. Similarly, when $\overline{E}_k^{13} = 0$, (7) can be checked directly because A_k^Δ and B_k^Δ are known.

With these two lemmas in place, we can now state and prove the main result of this section, which gives a convex upper bound on the guaranteed ℓ_2 semi-norm of a given system.

Theorem 5. *Assume that a system,* H^Δ*, has the realization* (2) *and the matrices* L *and* R *are given. Then* $\|LH^\Delta R\|_2^2 < \gamma \ \forall \Delta_k^{AB} \in \underline{\Delta}^{AB}, \ \Delta_k^{CD} \in \underline{\Delta}^{CD}$ *if* $\exists F_k^{AB} \in \underline{S}^{AB}, F_k^{CD} \in \underline{S}^{CD}, P_k, G_k, W_k$ *such that*

$$\gamma > \frac{1}{N} \sum_{k=1}^{N} \operatorname{tr} W_k \tag{8a}$$

$$\begin{bmatrix} F_k^{CD} & 0 & \overline{E}_k^{21} G_k & \overline{E}_k^{22} R & \overline{E}_k^{24} F_k^{CD} \\ \bullet & W_k & L\overline{C}_k G_k & L\overline{D}_k R & L\overline{E}_k^{23} F_k^{CD} \\ \bullet & \bullet & G_k + G_k^T - P_k & 0 & 0 \\ \bullet & \bullet & \bullet & I & 0 \\ \bullet & \bullet & \bullet & \bullet & F_k^{CD} \end{bmatrix} \succ 0 \tag{8b}$$

$$\begin{bmatrix} F_k^{AB} & 0 & \overline{E}_k^{11} G_k & \overline{E}_k^{12} R & \overline{E}_k^{14} F_k^{AB} \\ \bullet & P_{k+1} & \overline{A}_k G_k & \overline{B}_k R & \overline{E}_k^{13} F_k^{AB} \\ \bullet & \bullet & G_k + G_k^T - P_k & 0 & 0 \\ \bullet & \bullet & \bullet & I & 0 \\ \bullet & \bullet & \bullet & \bullet & F_k^{AB} \end{bmatrix} \succ 0 \tag{8c}$$

for $k = 1, \ldots, N$ *where* $P_{N+1} = P_1$.

Proof. First choose $\tau = 1$, perform the Cholesky factorizations

$$F_k^{AB} = R_k^{AB}\left(R_k^{AB}\right)^T$$
$$F_k^{CD} = R_k^{CD}\left(R_k^{CD}\right)^T$$

and note that

$$R_k^{AB} \in \underline{S}^{AB}$$
$$R_k^{CD} \in \underline{S}^{CD}.$$

We now consider two cases. If $L\overline{E}_k^{23} \neq 0$, we use Lemma 3 to conclude that

$$\begin{bmatrix} W_k & LC_k^{\Delta}G_k & LD_k^{\Delta}R \\ \bullet & G_k + G_k^T - P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0, \forall \Delta_k^{CD} \in \underline{\Delta}^{CD,u}\left(R_k^{CD}\right). \tag{9}$$

If $L\overline{E}_k^{23} = 0$, we note that since $LC_k^{\Delta} = L\overline{C}_k$ and $LD_k^{\Delta} = L\overline{D}_k$, the 2^{nd}, 3^{rd}, and 4^{th} rows and columns of (8b) are equivalent to (9). Thus, in either case, we conclude that (9) holds. Similarly,

$$\begin{bmatrix} P_{k+1} & A_k^{\Delta}G_k & B_k^{\Delta}R \\ \bullet & G_k + G_k^T - P_k & 0 \\ \bullet & \bullet & I \end{bmatrix} \succ 0, \forall \Delta_k^{AB} \in \underline{\Delta}^{AB,u}\left(R_k^{AB}\right). \tag{10}$$

Fixing $\Delta_k^{AB} \in \underline{\Delta}^{AB}$, $\Delta_k^{CD} \in \underline{\Delta}^{CD}$ and using Lemma 2 with (8a), (9), (10), and (3) concludes the proof. □

Before using Theorem 5, it should always be checked whether or not $\overline{E}_k^{13}$ and/or $L\overline{E}_k^{23}$ are nonzero. There are two reasons for this. For instance, suppose that $\overline{E}_k^{13} = 0$. In this case, using condition (8c) would introduce unnecessary conservatism into the analysis, i.e. our analysis would be less conservative if we instead directly checked (7). Second, from a computational standpoint, it makes more sense in this case to check (7) because that LMI has smaller dimensions and contains fewer variables than (8c). Similar considerations apply when $L\overline{E}_k^{23} = 0$. Thus, when $L\overline{E}_k^{23} = 0$, (8b) in Theorem 5 should be replaced by its 2^{nd}, 3^{rd}, and 4^{th} rows and columns. Similarly when $\overline{E}_k^{13} = 0$, (8c) in Theorem 5 should be replaced by its 2^{nd}, 3^{rd}, and 4^{th} rows and columns.

Output Feedback Controller Design

In this section, we apply the generalization of the Lyapunov shaping paradigm presented in [8] to controller design using the LMIs in Theorem 5. First, we let the uncertain LPTV plant, H^u, have the form

$$\begin{bmatrix} x_{k+1} \\ z_k^\infty \\ z_k \\ y_k \end{bmatrix} = \left[\begin{array}{c|ccc} A_k & B_k^1 & B_k^2 & B_k^3 \\ \hline C_k^1 & D_k^{11} & D_k^{12} & D_k^{13} \\ C_k^2 & D_k^{21} & D_k^{22} & D_k^{23} \\ C_k^3 & D_k^{31} & D_k^{32} & 0 \end{array}\right] \begin{bmatrix} x_k \\ w_k^\infty \\ w_k \\ u_k \end{bmatrix}$$

$$w_k^\infty = \Delta_k z_k^\infty, \quad \Delta_k \in \underline{\Delta}$$

where

$$\underline{\Delta} := \left\{ \text{diag}\left[\alpha^1 I_{m(1)}, \ldots, \alpha^{q'} I_{m(q')}\right] : \alpha^i \in [-1, 1] \right\}.$$

Now we let the controller have the form

$$\begin{bmatrix} x_{k+1}^K \\ u_k \end{bmatrix} = \left[\begin{array}{c|c} A_k^K & B_k^K \\ \hline C_k^K & D_k^K \end{array}\right] \begin{bmatrix} x_k^K \\ y_k \end{bmatrix}.$$

When the controller is brought inside the loop, the plant has the state space form

$$\begin{bmatrix} \overline{x}_{k+1} \\ z_k^\infty \\ z_k \end{bmatrix} = \left[\begin{array}{c|cc} \overline{A}_k & \overline{E}_k^{13} & \overline{B}_k \\ \hline \overline{E}_k^{x1} & \overline{E}_k^{x4} & \overline{E}_k^{x2} \\ \overline{C}_k & \overline{E}_k^{23} & \overline{D}_k \end{array}\right] \begin{bmatrix} \overline{x}_k \\ w_k^\infty \\ w_k \end{bmatrix}$$

$$w_k^\infty = \Delta_k z_k^\infty, \quad \Delta_k \in \underline{\Delta}$$

where the new state is given by

$$\overline{x}_k := \begin{bmatrix} x_k \\ x_k^K \end{bmatrix}$$

and the state space matrices are given by

$$
\begin{aligned}
\overline{A}_k &:= \begin{bmatrix} A_k + B_k^3 D_k^K C_k^3 & B_k^3 C_k^K \\ B_k^K C_k^3 & A_k^K \end{bmatrix} \\
\begin{bmatrix} \overline{E}_k^{13} & \overline{B}_k \end{bmatrix} &:= \begin{bmatrix} B_k^1 + B_k^3 D_k^K D_k^{31} & B_k^2 + B_k^3 D_k^K D_k^{32} \\ B_k^K D_k^{31} & B_k^K D_k^{32} \end{bmatrix} \\
\begin{bmatrix} \overline{E}_k^{x1} \\ \overline{C}_k \end{bmatrix} &:= \begin{bmatrix} C_k^1 + D_k^{13} D_k^K C_k^3 & D_k^{13} C_k^K \\ C_k^2 + D_k^{23} D_k^K C_k^3 & D_k^{23} C_k^K \end{bmatrix} \\
\begin{bmatrix} \overline{E}_k^{x4} & \overline{E}_k^{x2} \\ \overline{E}_k^{23} & \overline{D}_k \end{bmatrix} &:= \begin{bmatrix} D_k^{11} + D_k^{13} D_k^K D_k^{31} & D_k^{12} + D_k^{13} D_k^K D_k^{32} \\ D_k^{21} + D_k^{23} D_k^K D_k^{31} & D_k^{22} + D_k^{23} D_k^K D_k^{32} \end{bmatrix}.
\end{aligned}
$$

When the time-varying gain Δ_k is also brought inside the loop, it results in the realization in (2) with

$$
\begin{aligned}
\overline{E}_k^{11} &= \overline{E}_k^{21} = \overline{E}_k^{x1} \\
\overline{E}_k^{12} &= \overline{E}_k^{22} = \overline{E}_k^{x2} \\
\overline{E}_k^{14} &= \overline{E}_k^{24} = \overline{E}_k^{x4}.
\end{aligned}
$$

At this point, we apply the generalization of the Lyapunov shaping paradigm presented in [8]. This results in the following expressions for the terms in (8b) and (8c):

$$
\begin{aligned}
\overline{A}_k G_k &= \begin{bmatrix} A_k X_k + B_k^3 \hat{C}_k & A_k + B_k^3 \hat{D}_k C_k^3 \\ \hat{A}_k & Y_{k+1} A_k + \hat{B}_k C_k^3 \end{bmatrix} \\
\overline{B}_k &= \begin{bmatrix} B_k^2 + B_k^3 \hat{D}_k D_k^{32} \\ Y_{k+1} B_k^2 + \hat{B}_k D_k^{32} \end{bmatrix} \\
\overline{C}_k G_k &= \begin{bmatrix} C_k^2 X_k + D_k^{23} \hat{C}_k & C_k^2 + D_k^{23} \hat{D}_k C_k^3 \end{bmatrix} \\
\overline{D}_k &= \begin{bmatrix} D_k^{22} + D_k^{23} \hat{D}_k D_k^{32} \end{bmatrix} \\
\overline{E}_k^{13} &= \begin{bmatrix} B_k^1 + B_k^3 \hat{D}_k D_k^{31} \\ Y_{k+1} B_k^1 + \hat{B}_k D_k^{31} \end{bmatrix} \\
\overline{E}_k^{23} &= \begin{bmatrix} D_k^{21} + D_k^{23} \hat{D}_k D_k^{31} \end{bmatrix} \\
\overline{E}_k^{11} = \overline{E}_k^{21} &= \begin{bmatrix} C_k^1 X_k + D_k^{13} \hat{C}_k & C_k^1 + D_k^{13} \hat{D}_k C_k^3 \end{bmatrix} \\
\overline{E}_k^{12} = \overline{E}_k^{22} &= \begin{bmatrix} D_k^{12} + D_k^{13} \hat{D}_k D_k^{32} \end{bmatrix} \\
\overline{E}_k^{14} = \overline{E}_k^{24} &= \begin{bmatrix} D_k^{11} + D_k^{13} \hat{D}_k D_k^{31} \end{bmatrix} \\
G_k &= \begin{bmatrix} X_k & I \\ Z_k & Y_k \end{bmatrix} \\
P_k &= \begin{bmatrix} P_k^1 & P_k^2 \\ \bullet & P_k^3 \end{bmatrix}.
\end{aligned}
\tag{11}
$$

First note that the right-hand sides of all these equalities are affine in P_k^1, P_k^2, P_k^3, X_k, Y_k, Z_k, $\hat{A}_k$, $\hat{B}_k$, $\hat{C}_k$, $\hat{D}_k$. Also note that, W_k, F_k^{AB}, and F_k^{CD} remain unchanged after applying the Lyapunov shaping paradigm. Utilizing Theorem 5 with the expressions in (11) gives a set of matrix inequalities in the optimization variables W_k, P_k^1, P_k^2, P_k^3, X_k, Y_k, Z_k, $\hat{A}_k$, $\hat{B}_k$, $\hat{C}_k$, $\hat{D}_k$, F_k^{AB}, and F_k^{CD}. Of these variables, only P_k^1 and P_k^3 are symmetric. To reconstruct the controller from these variables, first choose M_k and N_k such that

$$N_k M_k = Z_k - Y_k X_k \tag{12}$$

and then substitute them into

$$\begin{bmatrix} A_k^K & B_k^K \\ C_k^K & D_k^K \end{bmatrix} = \begin{bmatrix} N_{k+1}^{-1} & -N_{k+1}^{-1} Y_{k+1} B_k^3 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \hat{A}_k - Y_{k+1} A_k X_k & \hat{B}_k \\ \hat{C}_k & \hat{D}_k \end{bmatrix} \begin{bmatrix} M_k^{-1} & 0 \\ -C_k^3 X_k M_k^{-1} & I \end{bmatrix}. \tag{13}$$

Note that if we fix F_k^{AB} and F_k^{CD}, the matrix inequalities in Theorem 5 become affine in the above optimization parameters. This means that minimizing the guaranteed ℓ_2 semi-norm performance of the system using this formulation is convex when these variables are fixed. Alternatively, when $\hat{B}_k, \hat{D}_k$, and Y_k are fixed, the matrix inequalities become affine in the above optimization parameters, again resulting in a convex optimization to minimize the guaranteed ℓ_2 semi-norm performance of the system. Based on these two facts, we can construct the following methodology for control design:

1. **Initial Controller Design:** Find a controller of the same order as the plant which achieves robust stability over some unstructured uncertainty set, $\underline{\Delta}(S_k)$. This should be done using mixed $\mathcal{H}_2/\mu$ synthesis, D-K iteration, or $\mathcal{H}_\infty$ optimal control. For certain plants, it may be appropriate to simply set the initial controller equal to 0.

2. **Initial Uncertainty Scalings:** First fix the controller found in the previous step and bring it inside the loop. Then use Theorem 5 to find permissible values of P_k, G_k, W_k, F_k^{AB}, and F_k^{CD}. (Do not use (11) in this step.)

3. **Control Design:** Fix the value of F_k^{AB} and F_k^{CD} found in the previous step and minimize the guaranteed ℓ_2 semi-norm performance using convex optimization applied to Theorem 5 with (11).

4. **Uncertainty Scaling:** Fix the values of $\hat{B}_k, \hat{D}_k$, and Y_k found in the previous step and minimize the guaranteed ℓ_2 semi-norm performance using convex optimization applied to Theorem 5 with (11).

5. **Check Stop Criterion:** Check a relevant stopping criterion. If it is not met, return to step 3.

6. **Reconstruct Controller:** Use (12) and (13) to reconstruct the controller.

In this methodology, there are two subtleties which a good implementation should exploit during a preprocessing phase. First, to reduce conservatism, it should be checked for each LMI at each time step whether or not the uncertainty scalings are necessary. For instance, if $LD_k^{21}, D_k^{31} = 0$, then $L\overline{E}_k^{23} = 0$ and the generalized Lyapunov shaping paradigm should be applied to only the 2^{nd}, 3^{rd}, and 4^{th} rows and columns in (8b) for that time index. Second, the coupling between variables should be checked at each time step. For instance, if $D_k^{31} = 0$, then F_k^{AB} and F_k^{CD} are decoupled from $\hat{B}_k$ and $\hat{D}_k$, i.e. in step 4 of the methodology above, it is only necessary to fix Y_k for that particular time index.

This methodology is similar to D-K iteration because in both, the controller design process alternates between finding a controller which optimizes performance and scaling the uncertainty to reduce conservatism in the uncertainty model. There is one benefit, however, that this methodology has over D-K iteration. Although it is necessary to fix the controller when scaling the uncertainty in D-K iteration, only some of the control parameters need to be fixed when scaling the uncertainty in this methodology. Moreover, in some cases, none of the control parameters need to be fixed in order to scale the uncertainty.

Although this paper does not consider multiobjective control design, all of the techniques presented here could be trivially extended to allow for control design with multiple guaranteed ℓ_2 semi-norm performance objectives. In particular, we would apply Theorem 5 to each objective, i.e. each choice of (L, R). In this case, since our controller reconstruction does not depend on the values on W_k, F_k^{AB}, F_k^{CD}, P_k^1, P_k^2, and P_k^3, these variables can be allowed to be constraint-dependent, i.e. each application of Theorem 5 with (11) could have different values for these variables.

Track-Following Control

Figure 1 shows the structure of a hard disk drive with dual-stage actuation. The model of our system in

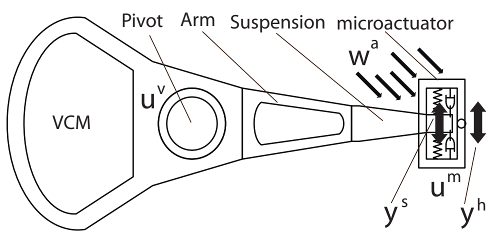

Figure 1: Schematic of the dual-stage hard drive

discrete time with uncertain parameters is given by

$$\begin{bmatrix} p(z) \\ y(z) \end{bmatrix} = H^{gen}(z) \begin{bmatrix} w(z) \\ u(z) \end{bmatrix} \tag{14}$$

where

$$p := \begin{bmatrix} y^h - r \\ y^h - y^s \\ u^v \\ u^m \end{bmatrix} \qquad w := \begin{bmatrix} w^r \\ w^a \\ n^{PES} \\ n^{RPES} \end{bmatrix}$$

$$y := \begin{bmatrix} y^h - r + n^{PES} \\ y^h - y^s + n^{RPES} \end{bmatrix} \quad u := \begin{bmatrix} u^v \\ u^m \end{bmatrix}$$

$$\begin{bmatrix} y^h(z) \\ y^s(z) \end{bmatrix} := \sum_{i=1}^{4} \begin{bmatrix} 1 & 0 \\ c^{1,i} & c^{2,i} \end{bmatrix} \left(zI - \begin{bmatrix} a^{1,i} & 1 \\ a^{2,i} & 0 \end{bmatrix} \right)^{-1} B^i \begin{bmatrix} w^a(z) \\ u^v(z) \\ u^m(z) \end{bmatrix}$$

$$a^{j,i} := \overline{a}^{j,i} + m^{j,i}\delta^i, \quad \delta^i \in [-1, 1]$$

$$r(z) := \left(\frac{2.355}{z - 0.9627} + \frac{0.557z + 0.5541}{z^2 - 1.984z + 0.9841} \right) w^r(z)$$

and the model parameters are summarized in Table 1. (Note that this model has three normalized uncertain-

Table 1: HDD model parameters

mode i	$\begin{bmatrix} \overline{a}^{1,i} \\ \overline{a}^{2,i} \end{bmatrix}$	$\begin{bmatrix} m^{1,i} \\ m^{2,i} \end{bmatrix}$	$\begin{bmatrix} c^{1,i} \\ c^{2,i} \end{bmatrix}$
1	$\begin{bmatrix} 1.9924 \\ -0.9925 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.9054 \\ -0.0946 \end{bmatrix}$
2	$\begin{bmatrix} 1.1818 \\ -0.9726 \end{bmatrix}$	$\begin{bmatrix} 0.1171 \\ -0.0071 \end{bmatrix}$	$\begin{bmatrix} 0.7883 \\ 6.5939 \end{bmatrix}$
3	$\begin{bmatrix} 0.4461 \\ -0.9605 \end{bmatrix}$	$\begin{bmatrix} 0.1995 \\ -0.0102 \end{bmatrix}$	$\begin{bmatrix} -1.9184 \\ 9.8265 \end{bmatrix}$
4	$\begin{bmatrix} 1.8188 \\ -0.8937 \end{bmatrix}$	$\begin{bmatrix} 0.0455 \\ -0.0299 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

mode i	B^i
1	$\begin{bmatrix} 0 & 0.4716 & 0 \\ 0 & 0.3212 & 0 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 0.1239 & 0 \\ 0 & -0.0655 & 0 \end{bmatrix}$
3	$\begin{bmatrix} -0.0678 & 0.0353 & 0 \\ 0.0317 & -0.0165 & 0 \end{bmatrix}$
4	$\begin{bmatrix} 0.1111 & -0.6329 & 1.5314 \\ -0.0209 & -0.2080 & 1.4749 \end{bmatrix}$

ties: δ^2, δ^3, and δ^4.) In this model, the elements of w represent the disturbances to our system (zero-mean white Gaussian noise with unit covariance); w^r, w^a, n^{PES}, and n^{RPES} respectively represent disturbances on the head position (see [5]), airflow disturbance, PES sensor noise, and RPES sensor noise. The PES measurement noise is relatively large in this model ($1nm$ at 1σ) because we are assuming that we are measuring the PES via a laser doppler velocimeter, as we will do when we try to implement the controller in the future. The elements of u represent the control inputs; u^v and u^m respectively represent the control inputs into the voice coil motor (V) and the MA (V). The elements of p and y respectively represent the signals we would like to keep small in closed loop and our measurements; y^h and y^s respectively represent the head displacement relative to the center of the data track (nm) and the the suspension tip displacement (nm).

In our final controller design, although we will measure the RPES and actuate our system at the rate of $50kHz$, we are bound by the constraint that the PES can only be measured at the rate of $25kHz$. Thus, as discussed in [6], the signal which should be sent to our controller is

$$\tilde{y}_k = \Omega_k y_k$$

where

$$\Omega_k = \begin{cases} \operatorname{diag}[1,1], & k \text{ odd} \\ \operatorname{diag}[0,1], & k \text{ even} \end{cases}.$$

Note that because Ω_k is a LPTV gain with period 2, it results in a LPTV system with period 2 when it is brought inside the model (14).

With this in place, we chose the output and input weights to respectively by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = I$$

and minimized the guaranteed ℓ_2 semi-norm performance of H^{gen} with the multi-rate sampler Ω_k using the above methodology. Table 2 shows the guaranteed costs of the closed loop system obtained by taking the square roots of the diagonal elements of $0.5(W_1+W_2)$. Note that because the RPES was not being penalized in the control design, no estimate of the worst case RPES was generated. To verify these bounds, we then

Table 2: Worst case RMS 1σ signal values, as reported by solver

Signal	PES (nm)	RPES (nm)	u^v (V)	u^m (V)
Value	5.0480	N/A	3.2536	1.8923

performed a Monte Carlo analysis of the ℓ_2 semi-norm of our uncertain closed loop system to each signal in y. This analysis consisted of two parts–examining the performance of the nominal system and examining the worst case performance (for each signal in y) over 400 random samples of the closed loop system. These results are shown in Table 3.

Table 3: Monte Carlo analysis of RMS 1σ signal values

	PES (nm)	RPES (nm)	u^v (V)	u^m (V)
Nominal	4.8133	9.7314	1.7252	1.8014
Worst Case	4.8969	10.4680	1.8538	1.8195

Conclusion

In this paper, we have proposed a new method for designing track-following controllers for HDDs which achieve robust performance. The control design methodology proposed, although it is not guaranteed to find a controller which locally minimizes guaranteed ℓ_2 semi-norm performance over the structured uncertainty set (i.e. it globally optimizes the guaranteed ℓ_2 semi-norm performance over some, slightly conservative, unstructured uncertainty set), it is conceptually similar to the D-K iteration heuristic for μ-synthesis which has been widely successful in many control design applications. However, unlike D-K iteration, the methodology presented here does not require all of the control parameters to be fixed in order to optimize the uncertainty scalings. This design methodology was used to design a HDD controller with multi-rate sampling and actuation characteristics which achieves robust performance.

Bibliography

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15 of *SIAM studies in applied mathematics*. Society for Industrial and Applied Mathematics, 1994.

[2] Richard Conway. Multi-objective control design for discrete time periodic systems via convex optimization. Technical report, University of California at Berkeley, 2007.

[3] S. H. Esfahani and I. R. Petersen. LMI approach to the output-feedback guaranteed cost control for uncertain time-delay systems. In *Proceedings of the IEEE Conference on Decision and Control*, volume 2, pages 1358–1363, December 1998.

[4] HGST. Perpendicular magnetic recording, 2006.

[5] X. Huang, R. Nagamune, and R. Horowitz. A comparison of multirate robust track-following control synthesis techniques for dual-stage and multi-sensing servo systems in hard disk drives. *IEEE Transactions on Magnetics*, 42(7):1896–1904, July 2006.

[6] R. Nagamune, X. Huang, and R. Horowitz. Multirate track-following control with robust stability for a dual-stage multi-sensing servo system in HDDs. In *Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference 2005*, pages 3886–3891, December 2005.

[7] K. Oldham, X. Huang, and R. Horowitz. Design, fabrication, and control of a high-aspect ratio microactuator for vibration suppression in a hard disk drive. In *Proceedings of the 16th IFAC World Congress*, July 2005.

[8] M. C. De Oliveira, J. C. Geromel, and J. Bernussou. Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems. *International Journal of Control*, 75(9):666–679, June 2002.

[9] I. R. Petersen, D. C. McFarlane, and M. A. Rotea. Optimal guaranteed cost control of discrete-time uncertain linear systems. *International Journal of Robust and Nonlinear Control*, 8(8):649–657, 1998.