Forces on a spherical particle with an arbitrary axis of rotation in a weak shear flow of a highly rarefied gas

Nan Liu and David B. Bogy

Computer Mechanics Laboratory, Department of Mechanical Engineering,

5146 Etcheverry Hall, University of California, Berkeley, CA 94720

(Dated: November 21, 2008)

Abstract

In this brief report, we derive analytical formulae for the force and torque on a spherical particle in a weak shear flow of a highly rarefied gas. The rotation axis of the particle is not restricted with regard to its direction. So this paper is an extension of [N. Liu and D. Bogy, Phys. Fluids **20**, 107102 (2008)] where the rotation axis is in the same direction as the gradient of the incoming shear flow. The contributions to the force from rotation of the particle and nonuniformity of the flow are shown to be decoupled. These two effects, however, do produce a coupled effect in the torque. Combined with the equation of motion, the trajectory of a spherical particle in a weak shear flow of a highly rarefied gas can be analyzed based on these analytical formulae together with appropriate initial conditions.

When a spherical particle moves in a shear flow of a rarefied gas, there exists three length scales: the radius of the particle R_0, the mean free path of the gas λ, and a length characterizing the shear strength G of the incoming flow $l_G = U_{f0}/G$ where U_{f0} is the speed of the particle relative to the flow at the center of the sphere, taken as a reference speed. Based on these three lengths, two Knudsen numbers can be defined: $\mathrm{Kn}_p = \lambda/R$ and $\mathrm{Kn}_G = \lambda/l_G$. When the radius of the sphere is much smaller then the mean free path, i.e., $\mathrm{Kn}_p \gg 1$, the flow, seen by the particle, is highly rarefied. By "weak shear flow" we mean that the shear strength G is so small that the other Knudsen number Kn_G is much smaller than 1. One application of this kind of problem is the motion of small particles in the gap between the slider and the disk in a hard disk drive. These particles may contact the slider and accumulate there, which increases the possibility of slider-induced damage of the disk and data loss. The flow in the gap, which is called the head disk interface (HDI), is set up by the disk moving at a speed of $10-30\mathrm{m/s}$ and is mainly a shear flow. For a large portion of the HDI, the gap spacing is around 1μm [1]. Given that the mean free path of air is 65nm, the Knudsen number Kn_G in those regions is on the order of 0.1. The particles are generated from different means and their sizes range from a few to hundreds of nanometers. For many slider designs, the gap spacing at the entrance of the HDI is less than 200nm, so only those particles with radius less than this value can enter the HDI. For smaller particles with size around 10nm, which are of concern here, the Knudsen number $\mathrm{Kn}_p \gg 1$.

The motion of a single particle in a fluid is usually calculated using Newton's second law [1, 2], and the essential part of this calculation is to obtain the forces on the particle. A widely known and documented result is the drag force on a sphere moving without rotation in a highly rarefied gas [3–7]. The first step to investigate general motion of particles is to consider the effect of the particle's rotation. Wang [8] studied the forces on a particle rotating around an axis perpendicular to the direction of the incoming *uniform* flow of a highly rarefied gas, and showed that the particle's rotation induces a lift force along the direction perpendicular to both the incoming flow and the axis of rotation. This lift force was found to be in the opposite direction from its counterpart in a continuum flow. Further studies considered nonuniformities of the incoming flow, among which a linear shear flow is the simplest. The linear shear flow is also a good approximation to a flow with its characteristic length scale much larger than the size of the particle, which underlies Saffman's explanation of the drift of small spheres in a Poiseuille flow based on his result for the lift force on a sphere in a linear shear flow of continuum fluid [9]. Kröger and Hütter [10] studied the forces on a sphere in a linear shear flow of a highly rarefied gas and qualitatively showed that an additional force is induced by the shear. Liu and Bogy [11] performed a quantitative study of this problem, and they derived a formula for the force on a sphere in a linear shear flow of a highly rarefied gas. They, however, restricted the analysis to the special case where the axis of rotation lies in the same direction as the gradient of the incoming shear flow. This brief report serves to extend Liu and Bogy's work to the general case where the particle's rotation direction is arbitrary with respect to the incoming flow. The primary results are analytical formulae of the force and torque on the sphere.

We first give an overview of the approach to be used here, which is the same as that adopted by Liu and Bogy [11]. Since the incoming flow, as seen by the particle, is highly rarefied, the interaction between the molecules coming to the particle and those reflected by the particle can be neglected. Then the force on a unit area of the surface of the particle becomes a linear combination of contributions from the incoming and reflected molecules. Due to the weakness of the shear strength of the incoming flow, the Knudsen number $\mathrm{Kn}_G \ll 1$ and the velocity distribution function for the incoming molecules can be obtained through the Chapmann-Enskog theory [4, 12]. The force on a unit area of the surface of the particle induced by the incoming molecules is then calculated through an integration based on this distribution function. The contribution from the reflected molecules is obtained from the boundary conditions relating the incoming and reflected molecules. Finally, the total force and torque on the particle are obtained by integrating the force on a unit area over the surface of the particle.

For the following analysis, two coordinate systems–one global and one local–have been set up and are shown in Fig. 1. In the global coordinate system $\{XYZ\}$, the axis X points in the flow direction while the axis Y points to the gradient of the shear flow, and the flow velocity is $U_f = U_{f0} + GY$. In the local coordinate system $\{xyz\}$ fixed to the sphere and located at (R_0, θ, ϕ), x is along the direction tangential to the parallel while y points to the center of the sphere. Here we allow for an arbitrary rotation of the sphere, so the axis of the rotation is not necessarily the same as the gradient of the incoming shear flow as in [11].

To get the force on a unit area of the surface of the sphere, we need the Chapman-Enskog velocity distribution function expressed in the local coordinate system. Let u, v, w be the components of the molecular velocity of the gas molecules in the local coordinate system,

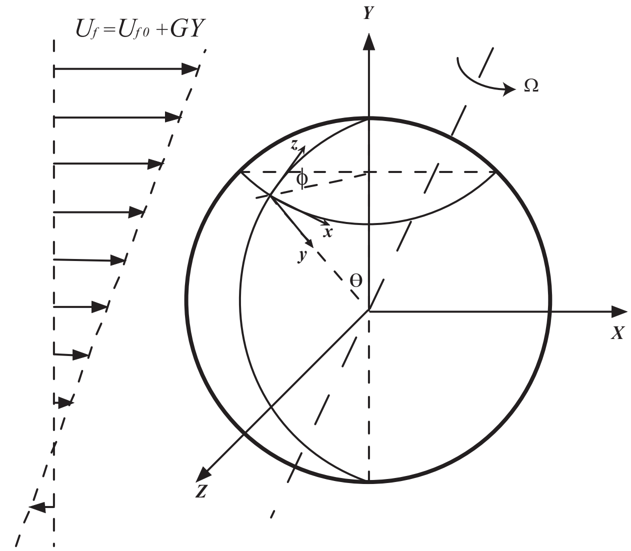

FIG. 1: The two coordinate systems set up for a spherical particle rotating at angular velocity Ω in a linear shear flow of a highly rarefied gas with the gradient of the shear being G. Note the axis of rotation is not necessarily parallel to the gradient

u', v', w' be the components of thermal velocity of the gas molecules in the local coordinate system, and let Ω_X, Ω_Y, Ω_Z be the components of the rotation vector in the global coordinate system. Then the velocity distribution function can be written as [4]

$$f = f_0 \left(1 + DU'V'\right). \tag{1}$$

Here,

$$f_0 = \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \exp\left\{-\beta^2[(u-u_0)^2 + (w-v_0)^2 + (w-w_0)^2]\right\}, \tag{2}$$

$$u_0 = U_{f0}\sin\phi - \Omega_Y R_0 \sin\theta - \Omega_X R_0 \cos\theta\cos\phi + \Omega_Z R_0 \cos\theta\sin\phi, \tag{3}$$

$$v_0 = U_{f0}\sin\theta\cos\phi, \tag{4}$$

$$w_0 = U_{f0}\cos\theta\cos\phi + \Omega_X R_0 \sin\phi + \Omega_Z R_0 \cos\phi, \tag{5}$$

$\beta = 1/\sqrt{2RT}$, R is the gas constant, T is the temperature of the particle and is assumed to be the same as the temperature at the far field, $D = -(5/4)\sqrt{\pi}\beta G\lambda$, and

$$U'V' = u'w'\sin\theta\sin\phi - u'v'\cos\theta\sin\phi + \frac{1}{2}(w'^2 - v'^2)\sin 2\theta\cos\phi - v'w'\cos 2\theta\cos\phi. \tag{6}$$

The only difference of Eq. (1) from the distribution function used in [11] is the exact form of u_0, v_0 and w_0, i.e., Eqs. (3)– (5). Thus the formulae for the forces on a unit area of the surface of the sphere that were derived in Ref. [11] and expressed as functions of u_0, v_0 and w_0 can be used as a basis for our further derivation. For self completeness of this report, we list these results here. In the local coordinate system, let p be the force along y direction and τ_x, τ_z be that along the x, z directions, respectively. Then

$$p = (2-\sigma_p)p_i + \sigma_p p_w, \tag{7}$$

$$\tau_x = \tau_{xi} - \tau_{xr} = \sigma_\tau \tau_{xi}, \tag{8}$$

$$\tau_z = \tau_{zi} - \tau_{zr} = \sigma_\tau \tau_{zi}, \tag{9}$$

and

$$\begin{aligned} p_i =& \rho \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} v^2 f \mathrm{d}u\mathrm{d}v\mathrm{d}w \\ =& \frac{\rho}{2\sqrt{\pi}\beta^2}\left\{\beta v_0 \mathrm{exp}(-\beta^2 v_0^2) + \sqrt{\pi}(\frac{1}{2}+\beta^2 v_0^2)[1+\mathrm{erf}(\beta v_0)]\right\} \\ &- \frac{\rho D}{4\beta^4}[1+\mathrm{erf}(\beta v_0)]\sin\theta\cos\theta\cos\phi, \end{aligned} \tag{10}$$

$$\begin{aligned} \tau_{xi} =& \rho \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} uv f \mathrm{d}u\mathrm{d}v\mathrm{d}w \\ =& \frac{\rho u_0}{2\sqrt{\pi}\beta}\left\{\mathrm{exp}(-\beta^2 v_0^2) + \sqrt{\pi}\beta v_0[1+\mathrm{erf}(\beta v_0)]\right\} - \frac{\rho D}{8\beta^4}[1+\mathrm{erf}(\beta v_0)]\sin\phi\cos\theta \\ &- \frac{\rho D}{4\sqrt{\pi}\beta^3}u_0\mathrm{exp}(-\beta^2 v_0^2)\sin\theta\cos\theta\cos\phi, \end{aligned} \tag{11}$$

$$\begin{aligned} \tau_{zi} =& \rho \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} wv f \mathrm{d}u\mathrm{d}v\mathrm{d}w \\ =& \frac{\rho w_0}{2\sqrt{\pi}\beta}\left\{\mathrm{exp}(-\beta^2 v_0^2) + \sqrt{\pi}\beta v_0[1+\mathrm{erf}(\beta v_0)]\right\} + \frac{\rho D}{8\beta^4}[1+\mathrm{erf}(\beta v_0)](\sin^2\theta-\cos^2\theta)\cos\phi \\ &- \frac{\rho D}{4\sqrt{\pi}\beta^3}u_0\mathrm{exp}(-\beta^2 v_0^2)\sin\theta\cos\theta\cos\phi, \end{aligned} \tag{12}$$

$$p_w = \frac{\rho}{4\beta^2}\left\{\mathrm{exp}(-\beta^2 v_0^2) + \sqrt{\pi}\beta v_0[1+\mathrm{erf}(\beta v_0]\right\} - \frac{\rho D}{8\beta^4}\mathrm{exp}(-\beta^2 v_0^2)\sin\theta\cos\theta\cos\phi, \tag{13}$$

where the quantities with a subscript i correspond to the contribution of the incoming flow, ρ is the density of air, $\mathrm{erf}(z) = (2/\sqrt{\pi})\int_0^z \mathrm{exp}(-t^2)dt$ is the error function, and σ_p and σ_τ are normal and tangential accommodation coefficients, which are allowed to be different.

Using Eqs. (7)– (13), we can get the total force and torque on the sphere by performing integration over the surface of the sphere. Here we need for this integration the relations Eq. (3)– (5) for transferring the above results in Eqs. (7)– (13) back into the global coordinate system, and we also use the identity

$$\int_0^{2\pi} \mathrm{erf}(b\cos\phi)\cos\phi \mathrm{d}\phi = \frac{2b}{\sqrt{\pi}} \int_0^{2\pi} \exp(-b^2\cos^2\phi)\sin^2\phi \mathrm{d}\phi, \tag{14}$$

where b is any function independent of ϕ. Since the speed of the flow is usually much less than the speed of sound, which is the case when the motion of particles in the HDI is of concern, higher order terms in βU_{f0} can be neglected. After a lengthy integration and retaining only terms up to the linear order in βU_{f0}, we obtain the force components $\{F_X, F_Y, F_Z\}$ on the sphere

$$\begin{aligned}
F_X =& \int_0^{2\pi}\int_0^{\pi} \left(p\sin\theta\cos\phi + \tau_x\sin\phi + \tau_z\cos\theta\cos\phi\right) R_0^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \\
=& \frac{1}{3}\rho\sqrt{2\pi RT}\left[4(2+\sigma_\tau-\sigma_p)+\pi\sigma_p\right] R_0^2 U_{f0}, && (15)\\
F_Y =& \int_0^{2\pi}\int_0^{\pi} \left(-p\cos\theta + \tau_z\sin\theta\right) R_0^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \\
=& -\frac{1}{6}(2+\sigma_\tau-\sigma_p)\pi\rho G R_0^2 \lambda U_{f0} + \frac{2}{3}\sigma_\tau \pi\rho\Omega_Z R_0^3 U_{f0}, && (16)\\
F_Z =& \int_0^{2\pi}\int_0^{\pi} \left(-p\sin\theta\cos\phi + \tau_x\cos\phi - \tau_z\cos\theta\sin\phi\right) R_0^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \\
=& -\frac{2}{3}\sigma_\tau\pi\rho\Omega_Y R_0^3 U_{f0}, && (17)
\end{aligned}$$

and the torque components $\{T_X, T_Y, T_Z\}$ on the sphere

$$\begin{aligned}
T_X =& \int_0^{2\pi}\int_0^{\pi} \left(\tau_x\cos\theta\cos\phi - \tau_z\sin\phi\right) R_0^3 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \\
=& -\frac{4}{3}\sigma_\tau\rho R_0^4\sqrt{2\pi RT}\Omega_X - \frac{1}{12}\sigma_\tau\rho\pi R_0^4\lambda G\Omega_Y, && (18)\\
T_Y =& \int_0^{2\pi}\int_0^{\pi} \tau_x R_0^3 \sin^2\theta \mathrm{d}\theta \mathrm{d}\phi \\
=& -\frac{4}{3}\sigma_\tau\rho R_0^4\sqrt{2\pi RT}\Omega_Y - \frac{1}{12}\sigma_\tau\rho\pi R_0^4\lambda G\Omega_X, && (19)\\
T_Z =& \int_0^{2\pi}\int_0^{\pi} \left(-\tau_x\cos\theta\sin\phi - \tau_z\cos\phi\right) R_0^3 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \\
=& -\frac{4}{3}\sigma_\tau\rho R_0^4\sqrt{2\pi RT}\Omega_Z + \frac{5(2\pi-1)}{48}\sigma_\tau\rho R_0^3\sqrt{2\pi RT}\lambda G, && (20)
\end{aligned}$$

or written in vector form

$$\begin{aligned}\mathbf{F} =&\frac{1}{3}\rho\sqrt{2\pi RT}\left[4(2+\sigma_\tau-\sigma_p)+\pi\sigma_p\right]R_0^2\mathbf{U}_{f0}-\frac{2}{3}\sigma_\tau\rho\pi R_0^3\mathbf{U}_{f0}\times\mathbf{\Omega}\\&-\frac{1}{6}(2+\sigma_\tau-\sigma_p)\rho\pi R_0^2U_{f0}\lambda\mathbf{G}\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{T} =&-\frac{4}{3}\sigma_\tau\rho R_0^4\sqrt{2\pi RT}\mathbf{\Omega}-\frac{1}{12}\sigma_\tau\rho\pi R_0^4\lambda\left(\mathbf{G}\cdot\mathbf{\Omega}\frac{\mathbf{U}_{f0}}{U_{f0}}+\frac{\mathbf{\Omega}\cdot\mathbf{U}_{f0}}{U_{f0}}\mathbf{G}\right)\\&+\frac{5}{48}(2\pi-1)\sigma_\tau\rho R_0^3\sqrt{2\pi RT}\lambda\frac{\mathbf{U}_{f0}\times\mathbf{G}}{U_{f0}}\end{aligned} \tag{22}$$

where $\mathbf{G}=G\mathbf{J}$ and $\mathbf{J}$ is the unit vector along the Y axis in the global coordinate system.

The first term in Eq. (21) is the drag force experienced by the particle. At first sight, this term looks different from the well-known result for the drag force on a sphere moving in a quiescent flow of a highly rarefied gas. Since we consider here the case in which the speed of the flow is much less than the speed of sound, only terms up to the linear order in βU_{f0} are retained, and the first term in Eq. (21) is just the reduced form of the general drag force formula with higher order terms in βU_{f0} neglected. The second term in Eq. (21) is the lift force induced by the rotation and the last term is the lift force induced by the nonuniformity of the flow. These lift forces have opposite directions from their counterparts in a continuum flow. For the case of a particle rotating about an axis perpendicular to the incoming uniform flow of a highly rarefied gas, i.e., $\Omega_Z=0$ and $G=0$, Eq. (21) reduces to Wang's result [8]. When the axis of rotation is in the same direction as the shear direction, i.e., $\Omega_X=\Omega_Z=0$, Eq. (21) reduces to the formula derived in [11]. The contributions of the shear and the rotation to the total force are decoupled, which is the same as in the special case studied by Liu and Bogy [11].

As in the continuum case [9], the torque $\mathbf{T}$ is independent of U_{f0}. However, due to the rarefaction of the gas, the torque here is proportional to the tangential accommodation coefficient σ_τ. The normal accommodation coefficient σ_p does not come into play since the normal force at any location on the surface of the sphere produces no torque. The first term in Eq. (22) is induced by the particle's rotation and is along the direction of the axis of rotation. A similar result exists for a particle rotating in a continuum flow, since, due to asymmetry of the flow field, the forces experienced by the upper half and lower half spheres are different, and torque arises. The last term in Eq. (22) is induced by the nonuniformity of the flow and points to a direction perpendicular to both the flow direction and the gradient of the flow. A particle moving in a linear shear flow of a continuum fluid experiences a similar torque, which is again induced by the asymmetry of the flow field around the sphere. The second term in Eq. (22) is due to the coupling effect of the shear and the rotation. This term is absent in the classical analysis of a particle rotating in a linear shear flow of a continuum fluid [9]. This kind of classical analysis is usually based on the linear Stokes equation, which is a reduced Navier–Stokes equation for the case when the flow velocity is so small or the viscosity is so large that the inertial effect can be neglected. Due to the linearity of the Stokes flow, the effects of rotation and shear are decoupled and no coupling terms similar to the second term in Eq. (22) appears. Lift force can not exist in such a Stokes flow analysis as well. In a classical paper analyzing the motion of a sphere in a weak shear flow of a continuum fluid, Saffman [9] used a perturbation method [13] and derived an analytical formula for the lift force on the sphere. Due to the complexity involved in this derivation, he did not carry out the study of the torque to the same level of approximation as the lift force. Since the Navier-Stokes equation itself is nonlinear, coupling between the shear and the rotation effects might also exist for the continuum case based on it rather than the Stokes approximation.

The terms solely due to the rotation effects, i.e., the first terms in Eq. (21) and (22), are independent of the Knudsen number while the other terms, involving $\mathbf{G}$, are all of first order in the Knudsen number Kn_G since the term λG can be written as $U_{f0}\lambda/(U_{f0}/G)$, or $U_{f0}\mathrm{Kn}_G$. This fact is consistent with our use of the Chapman-Enskog distribution function where only terms up to the linear order in Kn_G are retained.

Given the formulae Eqs. (21) and (22), the motion of particles in a shear flow of a highly rarefied gas, with appropriate initial conditions, can be determined from Newton's second law

$$m\frac{\mathrm{d}\mathbf{U}_{f0}}{\mathrm{d}t} = -\,\mathbf{F} \tag{23}$$

$$I_p\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} = \mathbf{T} \tag{24}$$

where t is the time and I_p is the moment of inertia of the particle.

In summary, the force and torque on a spherical particle in a weak shear flow of a highly rarefied gas are investigated in this report. Built upon previous results for the forces on a unit area of the surface of the sphere, we derive analytical formulae for the force and the torque, which include as special cases Wang's formula [8] for the force on a sphere rotating in a uniform flow of a highly rarefied gas and Liu and Bogy's formula [11] for the force on a sphere rotating around an axis restricted to be along the same direction as the gradient of the flow in a linear shear flow of a highly rarefied gas. The present formulae show that the coupling effect of the shear and rotation does not appear in the force but it is present in the torque. When the characteristic length scale of a general flow is much larger than the size of the particle, the flow can be locally approximated as a linear shear flow and the present formulae can then be used to calculate the force and torque on the sphere in this case as well. This knowledge of the force and torque on the sphere lays a foundation for the analysis of the motion of particles in the above mentioned cases.

The authors thank Computer Mechanics Laboratory, Department of Mechanical Engineering at University of California at Berkeley for supporting this research.

[1] S. Zhang and D. Bogy, Phys. Fluids **9**, 1265 (1997).

[2] A. Guha, Ann. Rev. Fluid Mech. **40**, 311 (2008).

[3] S. Schaaf and P. Chambré, *Flow of Rarefied Gases* (Princeton University Press, Princeton, 1961).

[4] G. Bird, *Molecular gas dynamics and the direct simulation of gas flows* (Oxford University Press, New York, 1994).

[5] Z. Li and H. Wang, Phys. Rev. E **68**, 061206 (2003).

[6] Z. Li and H. Wang, Phys. Rev. E **68**, 061207 (2003).

[7] E. Loth, AIAA J. **46**, 801 (2008).

[8] C. Wang, AIAA J. **10**, 713 (1972).

[9] P. Saffmann, J. Fluid Mech. **31**, 624 (1965).

[10] M. Kröger and M. Hütter, J. Chem. Phys. **125**, 044105 (2006).

[11] N. Liu and D. Bogy, Phys. Fluids **20**, 107102 (2008).

[12] L. Söderholm, Phys. Rev. E **66**, 031204 (2002).

[13] M. Van Dyke, *Perturbation methods in fluid mechanics* (The Parabolic Press, Stanford, 1975), annoted ed.