Lift forces on a sphere with slip boundary conditions in a weak shear flow

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ABSTRACT

Analytical formulae for drag and lift forces serve as a basis of simulating the motion of particles in a flow, which is required for the understanding and prediction of particle contamination, dispersion and deposition. In this paper, we use a perturbation method to solve the Navier-Stokes equation with the Maxwell slip boundary condition and obtain these formulae for spherical particles with slip boundary conditions, which act as a model for particles moving in a slightly rarefied gas as well as particles with hydrophobic and porous surfaces moving in a liquid. These formulae will facilitate our simulation of the motion of particles moving in a rarefied gas.

1 INTRODUCTION

Lift forces play an important role in understanding and numerically simulating the motion of particles in a non-uniform flow field, such as Poiseuille and Couette flows. These forces cause small particles to drift, i.e., move perpendicular to the streamline, which serves as a mechanism for a variety of phenomena involving the motion of particles, such as particle contamination on a slider flying over a moving disk in a hard disk drive [1], particle dispersion in a turbulent stream and deposition of particles–such as drugs–in the human respiratory tract, among others [2].

Two ways to simulate the motion of particles are Lagrange's and Euler's approaches, in which the particle's trajectory and the particle's density as a function of location are calculated, respectively [2]. Both approaches depend on analytical formulae for lift forces, which are currently only available for solid spheres with non-slip surfaces. This restriction does not hold when small particles are moving in a slightly rarefied gas, for which the mean free path, denoted by λ, is comparable to the characteristic size of the particle L. The Knudsen number, defined as $\mathrm{Kn} = \lambda/L$, in this case is not much less than 1. To capture this phenomenon, allowance should be made for velocity slip on the surface of the sphere, i.e. a velocity jump in the tangential direction between the velocity of the sphere and that of the flow field on the surface of the sphere. Velocity slip also appears on hydrophobic and porous surfaces [3]. Although the slip effects on drag force have been widely studied [4], little is known about their influence on lift forces [5]. Thus in the simulation of the motion of particles where velocity slip appears, the formula for the lift force without any slip effects is used together with the formula for the drag force with slip effects included. [1] [2]. This paper serves to remove this inconsistency by providing analytical formula for the lift forces also, which will facilitate the simulation of particles in the above mentioned cases.

In a non-uniform flow, lift forces on a solid particle are mainly induced by two mechanisms [5]. The first is related to the gradient of the flow. Saffman [6] investigated this mechanism in a weak shear flow by using a matched asymptotic expansion to include the inertial effects, which are neglected in creeping flow. His result shows that this lift force, also known as the Saffman force, is

$$F_{Saffman} = 6.46\mu U R^2 \sqrt{\frac{|k|\rho}{\mu}} \mathrm{sgn}(k) \tag{1}$$

where μ is the viscosity of the fluid, U is the velocity of the sphere relative to that of the fluid at the center of the sphere, R is the radius of the sphere, k is the gradient of the shear flow, ρ is the density of the fluid and $\mathrm{sgn}(\cdot)$ is the sign function. Saffman's result holds under the assumptions that $Re_k = \frac{|k|R^2\rho}{\mu} \ll 1$, $Re_p = \frac{UR\rho}{\mu} \ll 1$ and $Re_p \ll \sqrt{Re_k}$. McLaughlin [7] removed the last restriction by including the convection due to the fluid flow. He showed that the Saffman force in this case becomes

$$F_{Saffman} = \frac{9}{\pi}\mu U R^2 \sqrt{\frac{|k|\rho}{\mu}} J \mathrm{sgn}(k) \tag{2}$$

Here J is a complicated integral, which has been numerically integrated by McLaughlin [7], and it approaches 2.255 when Saffman's restrictions are satisfied. Legendre and Magnaudel [8] extended Saffman's and Mclaughlin's results to the case of a spherical bubble or droplet. They showed that Eq. (2) applies if a correction factor is added to it.

The second mechanism is related to the rotation of particles, and it was first investigated by Rubinow and Keller [9]. Their results can also be obtained in the framework of Saffman's approach [6]. This lift force is, however, one order smaller than the Saffman force for small particles with moderate angular velocities and, thus, is not considered here. In this paper, we follow Saffman's approach as well as later extensions of it to obtain formulae for the drag and lift forces as well as the torque on the sphere.

2 STATEMENT OF THE PROBLEM

We consider here a rotating sphere with angular velocity Ω in a weak shear flow, for which the velocity field, $\mathbf{u}$, satisfies

$$-\nabla p + \mu \nabla^2 \mathbf{u} = \rho \mathbf{u} \cdot \nabla \mathbf{u} \tag{3}$$

A coordinate system is set up with its origin lying at the center of the sphere. Its x axis

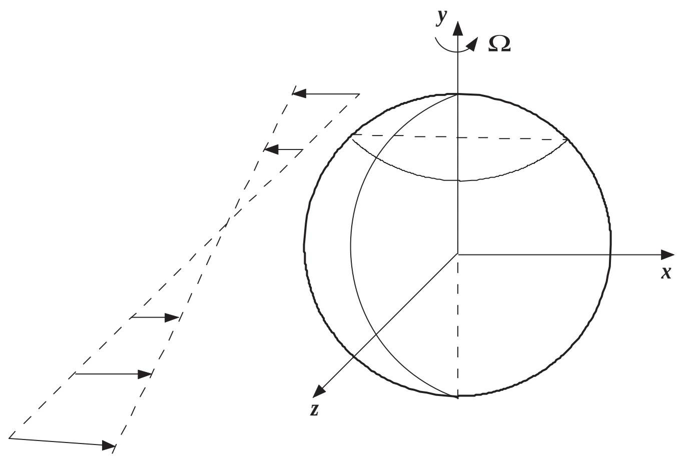

Figure 1: The coordinate system

points to the flow direction while the z axis coincides with the gradient of the flow field, as shown in Fig. 1.

The boundary condition remote from the sphere is that the influence of the presence of the sphere vanishes, i.e. the velocity $\mathbf{u}$ approaches $(U+kz)\mathbf{e_x}$ as $\mathbf{r}$ goes to infinity. On the surface of the sphere, the Maxwell slip boundary condition allowing for velocity slip is enforced. It requires that the velocity jump in the tangential direction be linearly proportional to the shear stress on the surface of the sphere [3] [10] [11], or

$$\beta(\mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}) = \boldsymbol{\tau} \cdot \mathbf{n} - (\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n})\mathbf{n} \quad \text{at} \quad r = R \tag{4}$$

where, in a slightly rarefied gas, β is inversely proportional to the mean free path of the air λ. The classical Maxwell slip boundary condition states that $\beta = \frac{\sigma}{2-\sigma}\frac{\mu}{\lambda}$ where σ is a constant related to the material property of the sphere, and it needs to be determined experimentally. For a hydrophobic or porous surface β is related to the slip length λ_s. The last boundary condition is the continuity of the normal velocity on the surface of the sphere:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{at} \quad r = R \tag{5}$$

As shown by Saffman [6], the force on the sphere can be expressed as

$$\mathbf{F} = -\int p\frac{\mathbf{r}}{r}dS - \int \frac{\mathbf{u}}{r}dS - r\frac{d}{dr}\int \frac{\mathbf{u}}{r}dS - \rho\int \mathbf{u}\frac{\mathbf{u}\cdot\mathbf{r}}{r}dS \tag{6}$$

where the integrals are performed on any spherical surface concentric to the sphere and with $r \geq R$. When the integrals are calculated on the surface of the sphere, i.e. with $r = R$, the last term vanishes due to the boundary condition Eq. (5).

Since the Reynolds number of interest here is quite small, the right hand side of Eq. (3), due to the inertial effect, is also very small. Thus the inertia will only introduce a small correction to the solution of corresponding creeping flow, which satisfies Eq. (3) with a zero right hand side. Therefore the solution can be expand in the order of Reynolds number, or, correspondingly, the inverse of the viscosity. Since the lift force is actually a first order effect, terms of an order higher than 1 are neglected in the expansion. We use superscripts (0) and (1) to denote zeroth and first order solutions. For example, the velocity is $\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}$.

Substituting the expansion of each quantity into Eq. (3), we find the zeroth order solution satisfies the equation of creeping flow and is subject to the above three boundary conditions. The solution is obtained here by using Lamb's general solution [10]. Following Lamb's original notation, the general solution of the creeping flow is

$$\mathbf{u}^{(0)} = (U + kz)\mathbf{e}_x + \frac{1}{\mu}\sum\left\{\frac{r^2}{2(2n+1)}\nabla p_n + \frac{nr^{2n+3}}{(n+1)(2n+1)(2n+3)}\nabla\frac{p_n}{r^{2n+1}}\right\} + \sum\left\{\nabla\phi_n + \nabla\chi_n\times\mathbf{r}\right\} \quad (7)$$

and

$$p^{(0)} = \sum\frac{p_n}{r^n} \quad (8)$$

Here p_n, ϕ_n and χ_n are solid spherical harmonics and satisfy

$$xu + yv + zw = \frac{1}{\mu}\sum\frac{nr^2p_n}{2(2n+3)} + \sum n\phi_n \quad (9)$$

$$x\xi + y\eta + z\zeta = \sum n(n+1)\chi_n \quad (10)$$

where $\{u, v, w\}$ and $\{\xi, \eta, \zeta\}$ are the velocity and vorticity of the flow, respectively. When the velocity remote from the sphere is used in Eq. (9)-(10), we find that only p_0, p_{-2}, p_{-3}, ϕ_{-2}, ϕ_{-3} and χ_{-2} need to be considered. The determination of these nonzero terms involves the application of the boundary conditions and is quite cumbersome. The final results are

$$p_{-2} = -\frac{3}{2}\mu UR\frac{2\mu+\beta R}{3\mu+\beta R}\frac{x}{r} \qquad p_{-3} = -\frac{36\mu^2R^3 + 11\mu\beta R^4 + 11\beta^2R^5}{\mu - \frac{5}{2}\beta R - \frac{\beta^2}{2\mu}R^2}\frac{kxz}{r}$$

$$\phi_{-2} = -\frac{1}{4}UR^3\frac{\beta R}{3\mu+\beta R}\frac{x}{r} \qquad \phi_{-3} = -\frac{12\mu R^5 + 6\beta R^6 + 5\frac{\beta^2}{\mu}R^5}{20\mu - 50\beta R - 5\frac{\beta^2}{\mu}R^2}\frac{kxz}{r}$$

$$\chi_{-2} = \frac{\beta R^4}{3\mu+\beta R}(\Omega - \frac{1}{2}k)\frac{y}{r^3}$$

The term p_0 is shown to be an undetermined constant, whose value is not of concern since we will only encounter its gradient in the following derivation.

Making use of Eq. (6), we find the drag force is

$$\mathbf{F}^{(0)} = \frac{2\mu + \beta R}{3\mu + \beta R} 6\pi\mu U R \mathbf{e}_x \tag{11}$$

and the torque is

$$\mathbf{T}^{(0)} = \frac{\beta R}{3\mu + \beta R} 8\pi R^3 (\frac{1}{2}k - \Omega)\mathbf{e}_y \tag{12}$$

Due to the linearity of the equation of creeping flow, no lift forces can exist at this order. Thus, we need to solve the first order system, which satisfies Eq. (3) with its right hand side replaced by $\mathbf{Q} = \rho\mathbf{u}^{(0)} \cdot \nabla\mathbf{u}^{(0)}$. As shown by Saffman [6], the non-zero integrals involved in Eq. (6) are related to $\mathbf{Q}$ by

$$\int p^{(1)} \frac{\mathbf{r}}{r} dS = \sum \frac{\mathbf{b}_n r^{n+1}}{(n+1)(n-2)} + \mathbf{A} + \mathbf{B} r^3 \tag{13}$$

$$\int \frac{\mathbf{u}^{(1)}}{r} dS = \sum \frac{\mathbf{a}_n r^{n+1}}{n(n+1)} + \sum \frac{\mathbf{b}_n r^{n+1}}{n(n+1)(n-2)} + \mathbf{C} r + \mathbf{D} + \frac{1}{2}\mathbf{B} r^3 \tag{14}$$

Here the terms containing $\mathbf{a}_n$ and $\mathbf{b}_n$ correspond to a particular solution of Eq. (3) with a non-zero right hand side $\mathbf{Q}$, and $\mathbf{a}_n$ and $\mathbf{b}_n$ are obtained from

$$\int \mathbf{Q} dS = \sum \mathbf{a}_n r^n \tag{15}$$

$$\int \mathbf{r} \nabla \cdot \mathbf{Q} dS = \sum \mathbf{b}_n r^n \tag{16}$$

The other terms in Eqs. (13)– (14) are general solutions of Eq. (3) with a zero right hand side and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are unknown constants to be determined by boundary conditions. These terms dominate the lift force due to a non-uniform flow field [6].

3 LIFT FORCE DUE TO A NONUNIFORM FLOW FIELD

Legendre and Magnaudet's solution, based on an order analysis, shows that the terms containing $\mathbf{a}_n$, $\mathbf{b}_n$ and $\mathbf{B}$ are at least one order smaller than the others [8]. Thus, the two integrals, after terms of smaller orders are neglected, become

$$\int p^{(1)}\frac{\mathbf{r}}{r}dS = \mathbf{A} \tag{17}$$

$$\int \frac{\mathbf{u}^{(1)}}{r}dS = \mathbf{C}r + \mathbf{D} \tag{18}$$

To get the lift force, we need to solve for the velocity field near the sphere and match it with the far field solution to determine $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$. This process is actually not necessary due to an observation first presented by Lengendre and Magnaudet [8]. The basic idea is that only a relation between $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ is needed since $\mathbf{C}$ can be determined from the far field solution, which itself is obtained from McLaughlin's results, as shown below. Since the particular solution containing $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ satisfies the same equation and boundary conditions on the sphere as the zeroth order solution, the relationship between $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ is obtained by a comparison with the zeroth order solution, for which,

$$\int p^{(0)}\frac{\mathbf{r}}{r}dS = -2\pi\mu UR\frac{2\mu+\beta R}{3\mu+\beta R}\mathbf{e}_x \tag{19}$$

$$\int \frac{\mathbf{u}^{(0)}}{r}dS = 4\pi U(r - \frac{2\mu+\beta R}{3\mu+\beta R}R)\mathbf{e}_x \tag{20}$$

Comparing Eqs. (17)– (20), we get

$$\mathbf{A} = -\frac{2\mu+\beta R}{3\mu+\beta R}\frac{\mu R}{2}\mathbf{C} \qquad \mathbf{D} = -\frac{2\mu+\beta R}{3\mu+\beta R}R\mathbf{C} \tag{21}$$

Thus using Eq. (21) in Eq. (6), we get

$$F_z = \frac{2\mu + \beta R}{3\mu + \beta R}\frac{3\mu R}{2}C_z \tag{22}$$

The determination of C_z requires solving for the velocity field far from the sphere, which satisfies

$$\rho(U + kz)\frac{\partial \mathbf{v}}{\partial x} + \rho k(\mathbf{v}\cdot\mathbf{e}_z)\mathbf{e}_x = -\nabla p + \mu\nabla^2\mathbf{v} - \mathbf{F}^0\delta(\mathbf{r}) \tag{23}$$

where $\mathbf{v} = \mathbf{u} - (U + kz)\mathbf{e}_x$. The effect of the presence of the sphere on the far field solution is included here as a point force opposite to the drag force on the sphere.

Due to the linearity of Eq. (23), C_z can be obtained from Eq. (23) based on McLaughlin's results and

$$C_z = \frac{1}{\pi^2}|\mathbf{F}^{(0)}|\sqrt{\frac{k\rho}{\mu}}J\mathrm{sgn}(k) \tag{24}$$

where J is an integral which approaches 2.255 when Saffman's restriction holds and has been numerically evaluated by McLaughlin.

From Eqs. (11), (22) and (24), we obtain the Saffman force

$$\mathbf{F}^{slip}_{saffman} = \frac{9}{\pi}\frac{(2\mu + \beta R)^2}{(3\mu + \beta R)^2}\mu R^2 U\sqrt{\frac{k\rho}{\mu}}J\mathrm{sgn}(k)\mathbf{e}_z \tag{25}$$

4 DISCUSSION

For a sphere moving in a slightly rarefied gas, the continuum theory with a first order slip boundary condition is only applicable for Knudsen numbers, corresponding to $\mu/\beta R$ here, up to 0.1, beyond which the kinetic theory is needed. But as a check of our results, we consider the case where the Knudsen number approaches infinity. In this case the problem is the same as that for a spherical gas bubble moving in a weak shear flow, which has already been studied by Lengendre and Magnaudet [8]. Our results show that the Saffman force is reduced to 4/9 of its value when no slip exists on the surface of the sphere, which agrees with the results of Lengendre and Magnaudet.

For a sphere with hydrophobic or porous surfaces, it is more convenient to express all of the results in terms of slip length $\lambda_s = \mu/\beta$:

$$\mathbf{F}_{drag} = 6\pi\mu U R \frac{2\lambda_s + R}{3\lambda_s + R}\mathbf{e}_x \tag{26}$$

$$\mathbf{F}^{slip}_{saffman} = \frac{9}{\pi}\frac{(2\lambda_s + R)^2}{(3\lambda_s + R)^2}\mu R^2 U\sqrt{\frac{k\rho}{\mu}} J\mathrm{sgn}(k)\mathbf{e}_z \tag{27}$$

$$\mathbf{T} = \frac{R}{3\lambda_s + R} 8\pi R^3 (\frac{1}{2}k - \Omega)\mathbf{e}_y \tag{28}$$

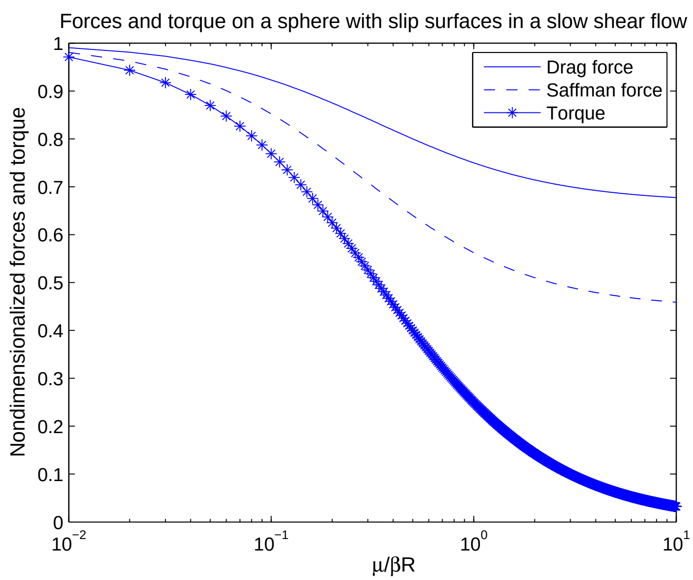

Figure 2: The drag and lift forces as well as the torque on a sphere with slip flow. Each quantity is nondimensionalized by its value with no slip. Kn $= \mu/\beta R$ for a sphere moving in a rarefied gas and Kn $= \lambda_s/R$ for a sphere with hydrophobic or porous surfaces.

Figure 2 shows how the forces and the torque change with $\mu/\beta R$ that corresponds to the Knudsen number for a sphere moving in a rarefied gas and λ_s/R for a sphere with hydrophobic or porous surfaces. Although the Knudsen number should be less than 0.1 for our formulae to apply, there is no limit on λ_s/R. Thus the abscissa in Fig. 2 goes from 0.01 to 10. Both the forces and the torque decrease with Knudsen number with the torque decreasing fastest and the drag force slowest. The drag force is the same as that for a stationary sphere lying in an otherwise uniform flow with velocity $\mathbf{U}$.

5 CONCLUSION

The lift force on a sphere with slip at its surface in a weak shear flow is investigated in this paper. The Maxwell slip boundary condition is used to model the velocity jump on the surface of the sphere. By solving the Navier-Stokes equation with this boundary condition, we obtain analytical formulae for the lift and drag forces as well as the torque. It is shown that the formulae for a sphere with no-slip at its surface apply when different correction factors are added for each formula. These obtained formulae serve as a basis for the calculation of the motion of particles in a slightly rarefied gas as well as particles with hydrophobic or porous surfaces in a liquid.

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