Shear force in a plane Poiseuille flow of a rarefied gas

Nan Liu and David B. Bogy

Computer Mechanics Laboratory, 5146 Etcheverry Hall

Department of Mechanical Engineering, University of California

Berkeley CA 94720

September 18, 2008

ABSTRACT

Poiseuille flow plays an important role in rarefied gas flows in microchannels with moving boundaries. Shear stress governs the stability of the boundaries and gas damping in these devices. In this paper, using some results from the linearized BGK-Boltzmann equation, we solve the conservation equations derived from the Boltzmann equation, and obtain an analytical formulae for the shear stress in a plane Poiseuille flow of a rarefied gas. It is shown that the value of the shear force on the boundaries is equal to the pressure gradient times half of the gas spacing. We also extend the classical definition of the Poiseuille number in an incompressible continuum flow to a compressible flow of a rarefied gas. Compared to the direct use of the classical definition, our definition makes experimental measurement and theoretical analysis easier for a compressible flow of a rarefied gas.

1 INTRODUCTION

Despite its simplicity, Poiseuille flow has some practical applications in the emerging field of microelectromechanical systems(MEMS) involving gas flows. One application is the gas lubrication in a hard disk drive, as shown in Fig. 1. The flow of the gas under a slider, which can be divided into the sum of Couette and Poiseuille flows [1] [2], serves as a lubrication layer and is set up by a disk moving at a speed around 10-20m/s. Other applications include, but are not limited to, MEMS filters for signal processing [3] and resonant sensors [4], where geometries similar to that shown in Fig. 1 exist. Due to the small size of the devices, these flows differ from the classical systems in that the gas is rarefied and the flows are compressible. Thus, they can not be analyzed in the framework of classical continuum fluid dynamics. Let us take the flow in the head disk interface(HDI), shown in Fig. 1, as an example. The gap spacing h in the HDI changes from several nanometers at the trailing end to near one micrometer. Since the mean free path of the air is around 65nm, the Knudsen number, defined as $\mathrm{Kn} = \lambda/h$, covers the whole range of values from $\mathrm{Kn} < 0.01$, corresponding to a continuum flow, to $\mathrm{Kn} > 10$, corresponding to a free molecular flow. The continuum theory, even supplemented with a slip boundary condition, only applies for Knudsen numbers less than 0.1, beyond which, the kinetic theory is needed.

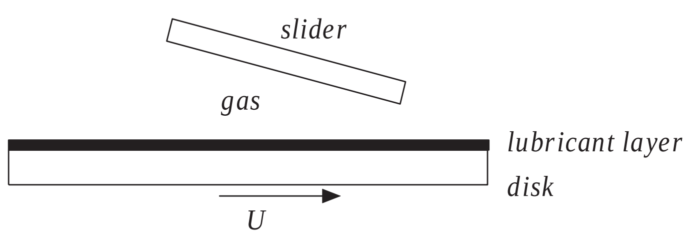

Figure 1: The geometry of the head disk interface, composed of a slider and a moving disk. A lubricant layer covering the disk serves to reduce impact of the slider on the disk. This figure is not to scale.

In the head disk interface, a layer of lubricant, with its thickness around 2nm, covers the disk and serves to reduce hard contacts between the slider and the disk. One major problem affecting the reliability of hard disk drives is the transfer of the lubricant from the disk to the slider, which can lead to instability and increases the possibility of slider-induced damage of the disk. Among other mechanisms contributing to this phenomenon, the shear-stress-induced deformation and instability of the lubricant is of major concern. Besides, the shear force, to a large extent, affects the motion and the stability of the slider, which is connected to a suspension not shown in Fig. 1. Furthermore, the shear stress also plays an important role in the heat transfer in the head disk interface of thermally actuated sliders, which determines the slider's protrusion induced by a temperature field set up by a heat generator embedded in the slider. This temperature-induced protrusion of the read/write element allows it to be closer to the disk while the bulk of the slider remains farther removed. It is currently used in the industry to increase the capacity of hard disk drives. In other laterally moving MEMS systems, the shear stress governs the gas damping, which affects the quality factor of these devices.

The shear stress in a flow such as the one shown in Fig. 1 is, to the leading order, a linear combination of the contributions from Couette and Poiseuille flows [1]. The Couette flow of a rarefied gas has been thoroughly investigated both analytically [5] [6] and numerically [7]. This, however, can not be said for a Poiseuille flow. The study of a Poiseuille flow of a rarefied gas was instigated by Kundsen's finding that a minimum exists in the mass flow rate when the gas becomes rarefied [8]. Thus much work on the Poiseuille flow is devoted to the study of the mass flow rate. The exact approach for addressing this problem is to solve the Boltzmann equation, which is very complicated and difficult to analyze. Instead, a linearized version of a model Boltzmann equation–known as the BGK-Boltzmann equation–is widely adopted. Cercignani [6] proposed a variational approach and obtained a semi-analytical result which confirms Knudsen's findings. Cercignani's approach and results were later used by Fukui and Kaneko [1] to derive the generalized Reynolds equation, from which the pressure distribution in the HDI is obtained. Sone [7] proposed a similarity solution to further simplify the linearized BGK-Boltzmann equation. His approach allows calculation of quantities other than the mass flow rate. Ohwada *et al.* [9] used Sone's idea to numerically solve the linearized Boltzmann equation and also demonstrated the existence of the minimum in the mass flow rate. The shear stress, however, has received little attention. Burgdorfer [10], among others, solved the Navier-Stokes equation supplemented with a first order slip boundary condition and got the shear stress in a near continuum flow. His result holds only for a Knudsen number less than 0.1. Bahukudumbi and Beskok [11] also solved the Navier-Stokes equation but with a phenomenological slip model claimed to be applicable for an arbitrary Knudsen number. The Navier-Stokes equation, however, is not applicable for a high Knudsen number [7]. Thus, the way to correctly address this problem is to use the Boltzmann equation or its equivalent. Valougeorgis [12] used a force balance approach. He showed that the value of the shear force is equal to $\frac{h}{2}\frac{\mathrm{d}p}{\mathrm{d}x}$ for any Knudsen number, where h is the gap spacing and $\frac{\mathrm{d}p}{\mathrm{d}x}$ is the pressure gradient along a direction parallel to the boundaries. This result agrees with that obtained by Bahukudumbi and Beskok [11]. Kang [13] solved the linearized BGK-Boltzmann equation and used Onsager's reciprocity relations to get the shear stress in a Poiseuille flow from the mass flow rate of a Couette flow. His result, however, shows that the value of the shear force, after it is nondimensionalized by $\frac{h}{2}\frac{\mathrm{d}p}{\mathrm{d}x}$, decreases and approaches a constant quickly as the Knudsen number increases. We note that Valougeorgis only considered the force balance between the pressure gradient and the shear stress. Due to the compressibility of the flow, the contributions from momentum flux and normal stress are nonzero, and thus need to be considered.

In this paper, our approach is to work with the conservation equations that are derived from the Boltzmann equation and are applicable for a rarefied gas. Based on some results from the linearized BGK-Boltzmann equation, we simplify the conservation equations and obtain an analytical formula for the shear stress upon considering the intrinsic symmetry of the Poiseuille flow.

This paper is organized as follows. We first present some preliminary results on the conservation equations and the linearized BGK-Boltzmann equation in section 2. An analytical formula for the shear stress in a Poiseuille flow of a rarefied gas is derived in section 3. In section 4, we propose a new definition of the Poiseuille number. The Poiseuille number is a nondimensionlized shear stress [14] that can be measured experimentally [15]. We discuss the results in section 5 and give a conclusion in section 6.

2 CONSERVATION EQUATIONS AND THE LINEARIZED BGK-BOLTZMANN EQUATION

2.1 CONSERVATION EQUATIONS

The Boltzmann equation describes a flow of a rarefied gas. Since no external forces exist in our problem, the Boltzmann equation for a steady flow becomes

$$\xi_i \frac{\partial f}{\partial x_i} = J(f, f) \tag{1}$$

where f is the velocity distribution function of the gas molecules and ξ_i is the molecular velocity. $J(f, f)$ is a complicated integral and its exact form is not of concern here.

The first three moments of Eq. (1) with respect to ξ_i give the conservation equations for a steady flow: conservation of mass, momentum and energy. When the velocity of the flow is much less than the average thermal velocity of the gas, which is of the same order as the speed of sound in the gas, the energy equation is, to the leading order, decoupled from the other two equations, and the flow can be regarded as isothermal [1]. Thus, to get the shear stress, we only need to deal with the conservation equations of mass and momentum, which, for a plane Poiseuille flow, are:

$$\frac{\partial}{\partial x}\left(\rho v_x\right) = 0 \tag{2}$$

$$\frac{\partial}{\partial x}\left(\rho v_x^2 + \sigma_{xx}\right) + \frac{\partial}{\partial y}\left(\sigma_{xy}\right) = 0 \tag{3}$$

where ρ is the density of the gas, (v_x, v_y) is the flow velocity, and σ_{xx} and σ_{xy} are components of the stress tensor.

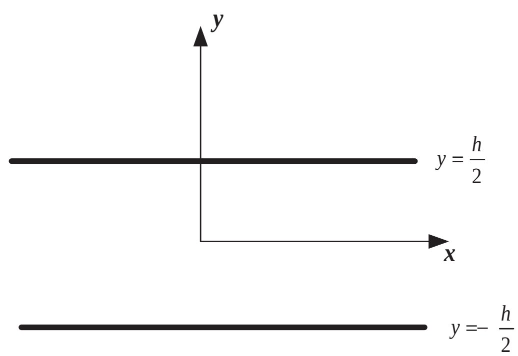

Figure 2: The geometry of a plane Poiseuille flow of a rarefied gas. The two boundaries lie at $y = -h/2$ and $y = h/2$

Using Eq. (2) in Eq. (3) and eliminating the derivatives of v_x, we get

$$-v_x^2 \frac{\partial \rho}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{4}$$

Further simplification of Eq. (4) requires some results from the linearized BGK-Boltzmann equation.

2.2 LINEARIZED BGK-BOLTZMANN EQUATION

In view of the complexity of the Boltzmann equation Eq. (1), Bhatnagar, Gross and Krook [16] proposed a model equation by using $\nu(f_e - f)$ to replace the right hand side of Eq. (1). Here ν

is a collision frequency that is related to the mean free path of the gas; f_e is a local Maxwellian with its parameters determined by f. Despite its simple form, the BGK-Boltzmann equation is still a nonlinear equation because of the appearance of f in f_e. When the velocity of the flow is much less than the average thermal velocity of the gas, the BGK-Boltzmann equation is linearized by assuming

$$f = f_0(1+\phi) \tag{5}$$

where f_0 is the Maxwellian distribution at a reference state:

$$f_0 = \frac{\rho_0}{(2\pi RT_0)^{3/2}} \exp\left(-\frac{\xi_i \xi_i}{2RT_0}\right)$$

where ρ_0, T_0 are the reference pressure and the reference temperature, and R is the specific gas constant.

Using Eq. (5) in the BGK-Boltzmann equation, i.e. Eq. (1) with its right hand side replaced by $\nu(f_e - f)$, and only retaining linear terms, we get the linearized BGK-Boltzmann equation for a steady flow of an isothermal gas

$$\xi_i \frac{\partial \phi}{\partial x_i} = \nu\left(-\phi - 1 + \frac{\rho}{\rho_0} + \frac{\xi_i v_i}{RT_0}\right) \tag{6}$$

The corresponding boundary condition accompanying Eq. (6), after it is linearized, is [7]

$$\phi(x_i, \xi_i) = (1-\alpha)\phi(x_i, \xi_i - 2\xi_j n_j n_i) - \alpha \frac{2\sqrt{\pi}}{(2RT_0)^2} \int_{\xi_k n_k < 0} \xi_j n_j \phi \exp\left(-\frac{\xi_k \xi_k}{2RT_0}\right) d\boldsymbol{\xi} \tag{7}$$

where α is the accommodation coefficient and n_i is the outward unit normal of the boundary.

Sone [7] showed that for a plane Poiseuille flow, a solution satisfying Eq. (6) and compatible with the boundary condition Eq. (7) can be expressed as

$$\phi = \frac{1}{p_0}\frac{\mathrm{d}p}{\mathrm{d}x}x + \frac{\xi_x}{\sqrt{2RT_0}}\phi_1(y,\xi_y,\xi_i\xi_i) \tag{8}$$

where ϕ_1 is an unknown function to be determined by Eq. (6) and the boundary condition Eq. (7).

From Eq. (8) and the kinetic theory [7]

$$\sigma_{xx} = -p_0 - \frac{2}{\pi^{3/2}}\frac{p_0}{(2RT_0)^{5/2}}\int \xi_x^2\phi \exp\left(-\frac{\xi_i\xi_i}{2RT_0}\right)d\boldsymbol{\xi} = -p_0 - \frac{\mathrm{d}p}{\mathrm{d}x}x \tag{9}$$

and σ_{xy} is nonzero. Thus the shear stress σ_{xy} is a first order effect and we only need to retain linear terms in Eq. (4) for an analysis of σ_{xy}.

3 SHEAR STRESS IN A POISEUILLE FLOW OF A RAREFIED GAS

In the framework of the linear theory, the first term in Eq. (4) is neglected and Eq. (4) reduces to

$$-\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial\sigma_{xy}}{\partial y} = 0 \tag{10}$$

Since Poiseuille flow is symmetric with respect to the centerline $y = 0$ as shown in Fig. 2, the shear forces on the two boundaries, τ_w, are equal to each other:

$$\tau_w = \sigma_{xy}n_y|_{y=-h/2} = \sigma_{xy}n_y|_{y=h/2}$$

The normal directions of the two boundaries are opposite to each other, i.e.

$$n_y|_{y=-h/2} = -n_y|_{y=h/2}$$

Thus,

$$\sigma_{xy}|_{y=-h/2} = -\sigma_{xy}|_{y=h/2} \tag{11}$$

Integrating Eq. (10) from $y = -h/2$ to $y = h/2$ and using Eq. (11), we get

$$\sigma_{xy}|_{y=h/2} = \frac{h}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{12}$$

This result is the same as that obtained through Valougeorgis's force balance approach. But our result shows that Eq. (12) holds even for a compressible flow of a rarefied gas when the velocity of the flow is much less than the average thermal velocity of the gas. Equation (12) can also be obtained from an integral form of the momentum equation, i.e. extending Valougeorgis's approach by including the shear stress σ_{xx} and momentum flux. But the linearized Boltzmann equation and the intrinsic symmetry of the Poiseuille flow are still needed for finally arriving at Eq. (12).

4 POISEUILLE NUMBER FOR A COMPRESSIBLE FLOW OF A RAREFIED GAS

For an incompressible flow of a continuum fluid, a natural way to nondimensionalize the shear stress is to use the Poiseuille number which, for a plane Poiseuille flow, is

$$Po = \frac{2h|\tau_w|}{\mu\overline{u}} \tag{13}$$

where τ_w is the shear force on the wall and is the same on the two walls, μ is the viscosity of the fluid, and $\overline{u}$ is the mean velocity.

For a compressible Poiseuille flow of a rarefied gas, we propose to define the Poiseuille number in a plane Poiseuille flow as

$$Po = \frac{2\rho h^2 |\tau_w|}{\mu q} \tag{14}$$

where q is the mass flow rate. In the calculation of the Poiseuille number for an incompressible flow in a tube with an arbitrary cross section, the mean velocity $\overline{u}$ in Eq. (13) is actually obtained from the mass flow rate [14]. Thus, Eq. (14), in which $\overline{u}$ is replaced by the mass flow rate q, is closer to the original spirit of using the Poiseuille number to nondimensionalize the shear stress. Besides, compared to the direct use of the definition in Eq. (13), our definition in Eq. (14) makes experimental measurement and theoretical analysis easier for a compressible flow of a rarefied gas since the mass flow rate q for a plane Poiseuille flow has already been thoroughly investigated. [2] [17].

From the kinetic theory, the viscosity is [18]

$$\mu = \frac{5}{32}\pi\rho\lambda\sqrt{\frac{RT}{2\pi}} \tag{15}$$

where λ is the mean free path.

Using Eqs. (12) and (15) in Eq. (14), we get

$$Po = \frac{32}{5}\frac{2}{\sqrt{\pi}}\frac{h}{\lambda}\frac{h^2\frac{\mathrm{d}p}{\mathrm{d}x}/\sqrt{2RT_0}}{q} = \frac{32}{5}\frac{D}{Q} \tag{16}$$

where $D = \frac{2}{\sqrt{\pi}}\frac{h}{\lambda}$ is the inverse Knudsen number and $Q = \frac{q}{h^2\frac{\mathrm{d}p}{\mathrm{d}x}/\sqrt{2RT_0}}$ is the nondimensionalized mass flow rate [19].

For diffusely reflected boundaries, which correspond to $\alpha = 1$ in Eq. (7), Cercignani [6] used a variational method to solve the BGK-Boltzmann equation with the boundary condition Eq. (7) and showed that

$$Q(D) = -\frac{1}{D} + \frac{C_{11} - \frac{D^2}{6}C_{12} + \frac{D^4}{144}C_{22}}{C_{11}C_{22} - C_{12}^2} \tag{17}$$

where

$$\begin{aligned}
C_{11} =& \frac{1}{\sqrt{\pi}}\left[8 - \frac{\sqrt{\pi}D^3}{12} + \frac{D^4}{16} - 2D(4+D^2)T_0(D)\right. \\
&\left. - \left(16 + 8D^2 + \frac{D^4}{8}\right)T_1(D) - D(16+D^2)T_2(D)\right] \\
C_{12} =& \frac{1}{\sqrt{pi}}\left[2 - \frac{\sqrt{\pi}D}{2} + \frac{D^2}{4} - 2DT_0D\right. \\
&\left. - \left(4 + \frac{D^2}{2}\right)T_1(D) - 2DT_2(D)\right] \\
C_{22} =& \frac{1}{\sqrt{\pi}}\left[1 - 2T_2(D)\right]
\end{aligned}$$

and $T_n(D)$ is the Abramowitz function defined as

$$T_n(D) = \int_0^\infty t^n \exp\left(-t^2 - \frac{D}{t}\right)dt$$

For a real surface, α is usually between 0.8 and 0.98 [20]. Fukui and Kaneko [2] studied the mass flow rate for a wide range of values of α by solving the linearized BGK-Boltzmann equation. We here make use of their results to show the influence of the accommodation coefficient on the Poiseuille number.

5 DISCUSSION

For a near continuum flow, which corresponds to an inverse Kundsen number $D \gg 1$, Sone [7] used an asymptotic approach to solve the linearized BGK-Boltzmann equation and showed that the mass flow rate of a plane Poiseuille flow between two diffusely reflected walls is asymptotically

$$Q(D) = \frac{D}{6} + 1.01619 + \frac{1.06528}{D} \tag{18}$$

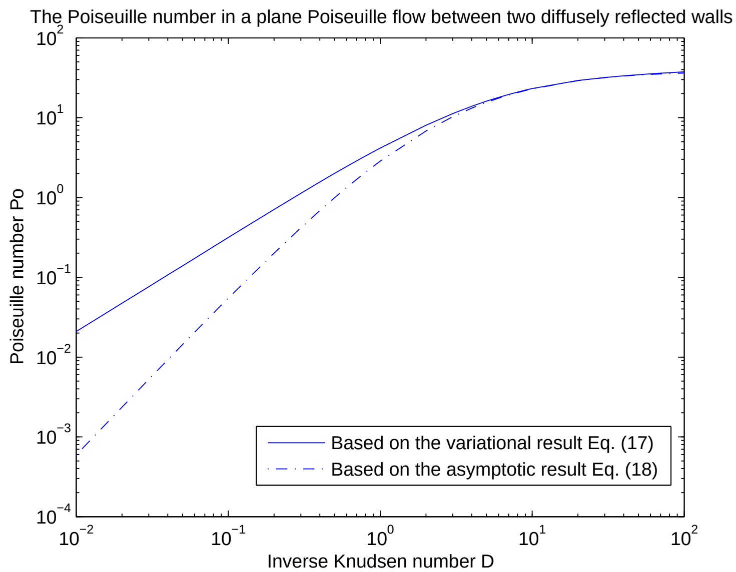

Figure 3: The Poiseuille number in a plane Poiseuille flow of a rarefied gas between two diffusely reflected walls.

We plot in Fig. 3 the dependence of the Poiseuille number Po on the inverse Knudsen number D in a plane Poiseuille flow between two diffusely reflected walls. The Poiseuille number increases with the inverse Knudsen number D and attains its maximum at $D \to \infty$, which corresponds to a continuum flow. Results based on Eq. (18) agree with those based on Eq. (17) for $D > 5$. For a lower D, the flow is no longer near continuum and Sone's result is not applicable.

Figure 4 shows a comparison of the Poiseuille numbers dependence on D in plane Poiseuille flows for three different accommodation coefficients of the walls: $\alpha = 1, 0.9$ and 0.8, respectively. We see that the accommodation coefficient only has a minor effect on the

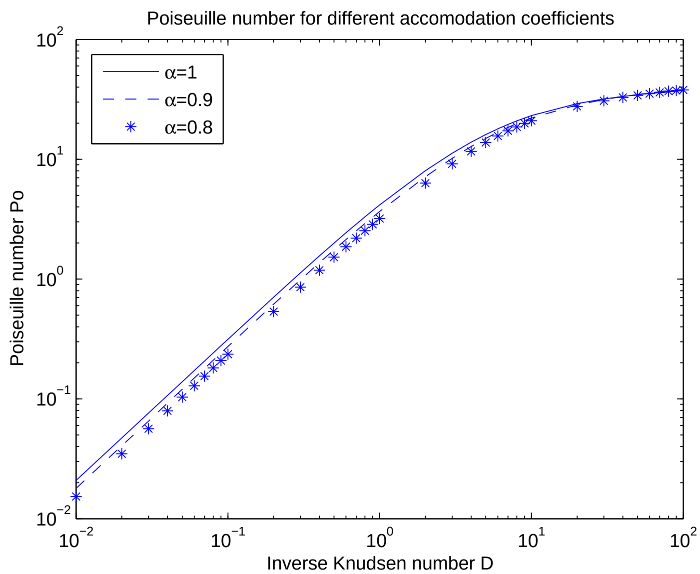

Figure 4: The influence of accommodation coefficient on the Poiseuille number in a plane Poiseuille flow of a rarefied gas.

Poiseuille number. Thus, Cercignani's variational result Eq. (17) can be used to estimate the Poiseuille number in a plane Poiseuille flow between two walls whose accommodation coefficients are different than 1.

In a flow such as the one shown in Fig. 1, the total shear stress is contributed by both the Couette and Poiseuille flows. Based on Sone's similarity solution, Fukui and Kaneko [1] showed that the total shear stress, at the leading order, is an addition of the contributions from Couette and Poiseuille flows. The contribution of the Couette flow has already been obtained by Liu and Lees [5] based on a moment method. Thus, the total shear forces on the two boundaries are

$$\tau_w\big|_{y=-\frac{h}{2}} = -\rho U \frac{RT_0}{2\pi}\frac{2\lambda}{2\lambda+h} - \frac{h}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{19}$$

$$\tau_w\big|_{y=\frac{h}{2}} = \rho U \frac{RT_0}{2\pi}\frac{2\lambda}{2\lambda+h} - \frac{h}{2}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{20}$$

Generally speaking, the pressure gradient changes with x. Thus the generalized Reynolds equation [1] needs to be solved for $\frac{\mathrm{d}p}{\mathrm{d}x}$ before Eqs. (19) and (20) can be used.

6 CONCLUSION

In this paper, we study the shear stress in a Poiseuille flow of a rarefied gas. Based on some results obtained from the linearized BGK-Boltzmann equation, we make use of an order analysis to simplify the conservation equations derived from the Boltzmann equation, and obtain an analytical formula for the shear stress. It is shown that the shear forces on the two boundaries are the same and are equal to $-\frac{h}{2}\frac{\mathrm{d}p}{\mathrm{d}x}$. This formula and that of the shear stress in a Couette flow give the total wall shear force in a flow of a rarefied gas in microchannels with a moving boundary, such as the head disk interface in a hard disk drive. These formulae serve as a basis for analyzing the stability of the slider and lubricant on the disk in a head disk interface as well as gas damping in some MEMS devices. A new definition of the Poiseuille number for a compressible flow of a rarefied gas is also proposed. It reduces to the classical one for an incompressible flow of a continuum fluid. Compared with the direct use of the classical Poiseuille number, this newly-defined Poiseuille number makes experimental measurement and theoretical analysis easier for a compressible flow of a rarefied gas.

ACKNOWLEDGEMENT

The authors thank Computer Mechanics Laboratory, Department of Mechanical Engineering at University of California at Berkeley for supporting this research.

References

[1] S. Fukui and R. Kaneko. Analysis of ultra-thin gas film lubrication based on linearized Boltzmann equation: first report–derivation of a generalized lubrication equation including thermal creep flow. *ASME Journal of Tribology*, 110:253–261, 1988.

[2] S. Fukui and R. Kaneko. A datebase for interpolation of poiseuille flow–rates for high Knudsen number lubrication problems. *ASME Journal of Tribology*, 112:78–83, 1990.

[3] L. Lin, R.T. Howe, and A.P. Pisano. Microelectromechanical filters for signal processing. *Journal of Microelectromechanical Systems*, 7:286–294, 1998.

[4] L. Li, G. Brown, and D. Uttamchandani. Air-damped microresonators with enhanced quality factor. *Journal of Microelectromechanical Systems*, 15:822–831, 2006.

[5] C.Y. Liu and L. Lees. Kinetic theory description of plane compressible couette flow. In L. Talbot, editor, *Rarefied Gas Dynamics*, pages 391–428, New York, 1961. Academic Press.

[6] C. Cercignani and C.D. Pagani. Variational approach to boundary-value problems in kinetic theory. *Physics of Fluids*, 9:1167–1173, 1966.

[7] Y. Sone. *Molecular gas dynamics: theory, techniques, and applications.* Birkahuser, Boston, 2006.

[8] E.H. Kennard. *Kinetic theory of gases, with an introduction to statistical mechanics.* McGraw-Hill, New York, 1938.

[9] T. Ohwada, Y. Sone, and K. Aoki. Numerical analysis of the shear and thermal creep flows of a rarefied gas over a plane wall on the basis of the linearized Boltzmann equation for hard-sphere molecules. *Physics of Fluids A*, 1:1588–1599, 1989.

[10] A. Burgdorfer. The influence of the molecular mean free path on the performance of hydrodynamic gas lubricated bearings. *ASME Journal of Basic Engineering*, 81:94–100, 1959.

[11] P. Buhukudumbi and A. Beskok. A phenomenological lubrication model for the entire Knudsen number. *Journal of Micromechanics and Microengineering*, 13:873–884, 2003.

[12] D. Valougeorgis. The friction factor of a rarefied gas flow in a circular tube. *Physics of Fluids*, 19:091702, 2007.

[13] S.C. Kang, R.M. Crone, and M.S. Jhon. A new molecular gas lubrication theory suitable for head-disk interface modeling. *Journal of Applied Physics*, 85:5594–5596, 1999.

[14] F.M. White. *Viscous fluid flow.* McGraw-Hill, New York, 2006.

[15] J.C. Harley, Y. Huang, H.H. Bau, and J.N. Zemel. Gas flow in micro-channels. *Journal of Fluid Mechanics*, 284:257–274, 1995.

[16] L. Bhatnagar, E. Gross, and M. Krook. A model for collision processes in gases. I. small amplitude processes in charged and neutral one-component systems. *Physical Review*, 94:511–525, 1954.

[17] C. Cercignani, M. Lampis, and S. Lorenzani. Plane poiseuille-couette problem in micro-electro-mechanical systems applications with gas-rarefaction effects. *Physics of Fluids*, 18:087102, 2006.

[18] G.A. Bird. *Molecular gas dynamics and the direct simulation of gas flows.* Oxford University Press, New York, 1994.

[19] C. Cercignani and A. Daneri. Flow of a rarefied gas between two parallel plates. *Physics of Fluids*, 34:3509–3513, 1963.

[20] G.S. Springer. Heat transfer in rarefied gases. *Advances in Heat Transfer*, 7:163–218, 1971.