Forces on a rotating particle in a shear flow of a highly rarefied gas

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ABSTRACT

The determination of the forces on a particle is required for the simulation of the particle's motion, which in turn is necessary for the simulation of particle contamination in a hard disk drive. In this paper, the forces on a rotating sphere in a shear flow of a highly rarefied gas are investigated analytically. The Chapman-Enskog distribution function is used to describe the molecules in the shear flow and a Maxwell-type boundary condition is assumed on the surface of the sphere. Expressions are obtained for the drag force as well as lift forces for the special case where the gradient of the shear flow is along the same direction as the axis of the particle's rotation. The effects of particle's rotation and the shear flow are shown to be decoupled. These lift forces also turn out to be in the opposite directions from their corresponding forces when the fluid is modeled as a continuum.

1 INTRODUCTION

Particle contamination is an important issue for the performance of a slider flying over a moving disk in a hard disk drive(HDD), the geometry of which is shown in Fig. 1. To understand this phenomenon, we need to know the particle's motion inside the gap between the slider and the disk, which is called the "head disk interface", or HDI. Due to the low volume density of particles in the HDI, the interaction between particles can be neglected. The essential part of this calculation is the force on a particle, from which the particle's trajectory can be calculated based on Newton's second law [1]. In our case, the particle's size is comparable to the mean free path of air, which is around 65nm, about 10 times the minimum spacing in the HDI of current HDDs. Thus, gas rarefaction effects need to be considered in the calculation of the forces.

In continuum theory, forces on a particle moving in an infinite fluid medium are well documented. Stokes first calculated the drag force on a spherical particle moving at low Reynolds number, i.e. $Re_p = \frac{U_{f0} R_0 \rho}{\mu} \ll 1$, where ρ is the density of the fluid, μ is viscosity of the fluid, $\mathbf{U}_{f0}$ is the velocity of the sphere relative to the fluid and R_0 is the radius of the sphere. Stokes' drag force is

$$\mathbf{F}_{Drag} = -6\pi\mu R_0 \mathbf{U}_{f0} \tag{1}$$

Oseen later extended Stokes' formula to higher Reynolds numbers by partially considering the inertial effect. His results were finally validated by Proudman and Pearson using perturbation methods [2] [3]. The lift force on a rotating particle in an infinite medium, known as Magnus force, at low Reynolds number was investigated by Rubinow and Keller [4], whose result shows that the force, under the condition $Re_\Omega = \frac{\Omega R_0^2 \rho}{\mu} \ll 1$, is

$$\mathbf{F}_{Magnus} = \pi R_0^3 \rho \mathbf{\Omega} \times \mathbf{U}_{f0} \tag{2}$$

where $\mathbf{\Omega}$ is the angular velocity of the particle. Saffmann [5] pioneered the work on lift forces on a particle moving in a strong linear shear flow at low Reynolds number, i.e. $Re_G = \frac{|G|R_0^2\rho}{\mu} \ll 1$ and $Re_p \ll \sqrt{Re_G}$, where G is the gradient of the shear flow. By using a matched asymptotic expansion, he showed that the lift force is

$$F_{Saffman} = 6.46\mu u R^2 \sqrt{\frac{|G|\rho}{\mu}} \tag{3}$$

The restriction of strong shear flow was later removed by McLaughlin [6], whose results are also subject to the restriction $Re_p \ll 1$ and $Re_G \ll 1$. Since the particle size we are considering is on the order of 100nm and the fluid velocity is about 10m/s, the Reynolds number here is also very small. On the other hand, the gradient of the shear flow set up by the moving disk is quite large due to the small gap in the HDI, which is on average less than $1\mu m$. Thus Saffman's assumption is valid here, and Eq. (3) would be applicable if the gas could be regarded as a continuum. In [7], Loth reviewed some recent progress in this field and also provided comparisons between experimental and analytical results.

Basset first considered the gas rarefaction effect on the drag force on a particle by using Stokes' approach but with the Maxwell slip boundary condition on the surface of the particle. However, his result is only applicable for a slightly rarefied gas. Cercignani generalized Basset's result by solving the linearized BGK-Boltzmann equation using a variational approach. His result is applicable for a sphere moving in an arbitrarily rarefied gas and agrees with experiments [8] [9]. A simple empirical formula was also provided by Sherman based on an interpolation of experimental results [10].

Despite the fact that much work has been done on the drag force on a particle moving in a rarefied gas, the lift force has received little attention, primarily due to the difficulty involved in solving the full Boltzmann equation. Most, if not all, available results are obtained for a highly rarefied gas. Wang [11] first calculated the lift force on a rotating sphere in a highly rarefied gas. He showed that the direction of the lift force is opposite to what it would be when the rarefaction effect is absent. Ivanov and Yanshin [12] extended Wang's work to a symmetric body, which includes a sphere as a special case. These results were rediscovered recently by Borg et al. [13] and Weidmann and Herczynski [14]. Using the same approach as Wang, Söderholm [15] and Borg et al. [16] calculated the lift force on a particle moving in a shear flow of a highly rarefied gas. They concluded that the lift force vanishes when the particle is a sphere. However, Kröger and Hütter's result showed that the lift force in this case is not zero and is in the opposite direction of the Saffmann force Eq. (3), although they did not give an explicit formula for the lift force [17].

In this paper, we model the particle as a sphere and consider a more general case than a fixed particle lying in a linear shear flow. We allow for the particle's rotation but assume that the axis of rotation is the same as the gradient of the shear flow. The main goal here is to get force formulae on a rotating particle in a shear flow of a highly rarefied gas. These formulae will enable us to calculate the particle's trajectory in the HDI. The paper is organized as follows. In section 2, the problem is formulated and all of the assumptions are stated. In section 3, the force on a unit area at some location on the surface of a spherical particle is calculated, which is then used to get the total force in section 4. A conclusion is given in section 5.

2 STATEMENT OF THE PROBLEM

Although the particles of interest may be of different shapes, we model them here as spheres. We also assume that the gas is highly rarefied, or is a free molecular gas. Under this assumption, the interaction between the incoming molecules and those reflected by a sphere is neglected. Thus we can get the velocity distribution function of the molecules neglecting the presence of the sphere. For a shear flow, the above velocity distribution function can be obtained from the Chapman-Enskog theory [18]:

$$f = f_0 \left[1 - B(C_i'C_j' - \frac{1}{3}C'^2\delta_{ij})\frac{\partial C_{0i}}{\partial x_j} \right] \tag{4}$$

where f_0 is the equilibrium Maxwellian distribution function, C is the total molecular velocity, C_0 is the mass average velocity, C' is the thermal velocity and B is a function of C' and the temperature T.

The present problem is defined as shown in Fig. 2. Here a sphere is rotating at the angular velocity Ω and lying in a shear flow with gradient G. A global coordinate system $\{XYZ\}$ fixed to the sphere is established with X pointing in the direction of the incoming shear flow, and Y is the axis of rotation. We assume the gradient of the shear flow is along the Y direction as well. U, V, W are used to denote the velocity components along the X,Y and Z directions, respectively. The velocity of the shear flow is $U_f = U_{f0} + GZ$, where U_{f0} is the velocity of the center of the sphere relative to the fluid flow and G is the gradient of the shear flow.

Since the gradient of the flow field is along the Y direction, the general Chapman-Enskog distribution, Eq. (4), in the present case, becomes

$$f = f_0 \left(1 + DU'V'\right) \tag{5}$$

where $f_0 = \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \exp\left\{-\beta^2[U'^2 + V'^2 + W'^2]\right\}$, $D = -\frac{5}{4}\sqrt{\pi}\beta^3 G\lambda$, $\beta = \frac{1}{\sqrt{2RT}}$, λ is the mean free path of air, U_0, V_0, W_0 are mass average velocity components, and $U' = U - U_0, V' = V - V_0, W' = W - W_0$ are the thermal velocity components. Note that the mass average velocity is different from the velocity of the shear flow since the coordinate system is fixed to the sphere, which is itself rotating with an angular velocity Ω.

3 FORCES ON A UNIT AREA ON THE SURFACE OF THE SPHERE

To calculate the forces on a unit area at a specific location on the surface of the sphere, we set up a local coordinate system $\{xyz\}$, as shown in Fig. 2. The y direction points to the center of the sphere while the x and z directions are tangential to the parallel and meridian, respectively. Let u, v, w denote the velocity components in this local coordinate system. Then the Chapmann-Enskog distribution, when expressed in terms of u, v, w is of the same form as Eq. (5), but now

$$U'V' = u'w' \sin\theta \sin\phi - u'v' \cos\theta \sin\phi + \frac{1}{2}(w'^2 - v'^2) \sin 2\theta \cos\phi - v'w' \cos 2\theta \cos\phi \tag{6}$$

and

$$f_0 = \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \exp\left\{-\beta^2[(u-u_0)^2 + (w-v_0)^2 + (w-w_0)^2]\right\} \tag{7}$$

where $u_0 = U_{f0} \sin\phi - \Omega R_0 \sin\theta, v_0 = U_{f0} \sin\theta \cos\phi$ and $w_0 = U_{f0} \cos\theta \cos\phi$.

According to the kinetic theory [18] [19], the normal force along the y direction is

$$p = p_i + p_r = (2 - \sigma_p)p_i + \sigma_p p_w \tag{8}$$

and the shear forces along the x and z directions are

$$\tau_x = \tau_{xi} - \tau_{xr} = \sigma_\tau \tau_{xi} \tag{9}$$

$$\tau_z = \tau_{zi} - \tau_{zr} = \sigma_\tau \tau_{zi} \tag{10}$$

In the above expressions, $p_i, \tau_{xi}, \tau_{zi}$ are due to the incoming molecules while $p_r, \tau_{xr}, \tau_{zr}$ are contributed by the molecules reflected by the sphere. p_w is the pressure due to the outgoing molecules when they all obey the equilibrium Maxwellian distribution function f_0 at the wall temperature. Here we assume the sphere is thermally highly conductive and of uniform temperature T_w, which is assumed to be the same as the temperature T_∞ at infinity. $\sigma_p = \frac{p_i - p_r}{p_i - p_w}$ and $\sigma_\tau = \frac{\tau_i - \tau_r}{\tau_i}$ are accommodation coefficients, which represent the percent of incoming molecules that are diffusely reflected, or accommodated to a Maxwellian distribution at the wall temperature. Here we allow for different accommodation coefficients for normal and shear forces.

Based on the assumption of a highly rarefied gas [19],

$$\begin{aligned} p_i &= \rho \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} v^2 f \mathrm{d}u \mathrm{d}v \mathrm{d}w \\ \tau_{xi} &= \rho \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} uvf \mathrm{d}u \mathrm{d}v \mathrm{d}w \\ \tau_{zi} &= \rho \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} wvf \mathrm{d}u \mathrm{d}v \mathrm{d}w \end{aligned} \tag{11}$$

where the lower limit for v is 0 since we are considering only the incoming molecules.

Using the Chapman-Enskog distribution function expressed in the local coordinate system, we can integrate the above equations to get the normal and shear forces. The integration is actually performed with u', v', w' and the results are

$$\begin{aligned} p_i = &\frac{\rho}{2\sqrt{\pi}\beta^2} \left\{ \beta v_0 \exp(-\beta^2 v_0^2) + \sqrt{\pi}(\frac{1}{2} + \beta^2 v_0^2)[1 + \mathrm{erf}(\beta v_0)] \right\} \\ &- \frac{\rho D}{4\beta^4}[1 + \mathrm{erf}(\beta v_0)] \sin\theta \cos\theta \cos\phi \end{aligned} \tag{12}$$

$$\begin{aligned} \tau_{xi} = &\frac{\rho U_{f0}}{2\sqrt{\pi}\beta} \left(\sin\phi - \frac{\Omega R_0}{U_{f0}} \sin\theta \right) \left\{ \exp(-\beta^2 v_0^2) + \sqrt{\pi}\beta v_0 [1 + \mathrm{erf}(\beta v_0)] \right\} \\ &- \frac{\rho D}{8\beta^4}[1 + \mathrm{erf}(\beta v_0)] \sin\phi \cos\theta - \frac{\rho D}{4\sqrt{\pi}\beta^3} u_0 \exp(-\beta^2 v_0^2) \sin\theta \cos\theta \cos\phi \end{aligned} \tag{13}$$

$$
\begin{aligned}
\tau_{zi} =& \frac{\rho w_0}{2\sqrt{\pi}\beta}\left\{\exp(-\beta^2 v_0^2) + \sqrt{\pi}\beta v_0[1+\mathrm{erf}(\beta v_0)]\right\} \\
&+ \frac{\rho D}{8\beta^4}[1+\mathrm{erf}(\beta v_0)](\sin^2\theta - \cos^2\theta)\cos\phi - \frac{\rho D}{4\sqrt{\pi}\beta^3}u_0 \exp(-\beta^2 v_0^2)\sin\theta\cos\theta\cos\phi
\end{aligned}
\tag{14}
$$

where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}dt$ is the error function.

Based on the previous discussion, p_w denotes the pressure due to the outgoing molecules as if they all obey the Maxwellian velocity distribution function at the wall temperature. The contribution from every molecule of this kind is

$$
\frac{p_w}{N_w} = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} v^2 f_0 \mathrm{d}u\mathrm{d}v\mathrm{d}w}{\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} f_0 \mathrm{d}u\mathrm{d}v\mathrm{d}w}
\tag{15}
$$

where N_w is the number of molecules. From the conservation of the number of molecules, N_w is equal to the total number of incoming molecules:

$$
N_w = n\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} v f \mathrm{d}u\mathrm{d}v\mathrm{d}w
\tag{16}
$$

where n is the volume density of molecules. Thus

$$
\begin{aligned}
p_w &= N_w\frac{m\sqrt{\pi}}{2\beta} \\
&= \frac{\rho}{4\beta^2}\left\{\exp(-\beta^2 v_0^2) + \sqrt{\pi}\beta v_0[1+\mathrm{erf}(\beta v_0]\right\} - \frac{\rho D}{8\beta^4}\mathrm{e}^{-\beta^2 v_0^2}\sin\theta\cos\theta\cos\phi
\end{aligned}
\tag{17}
$$

4 RESULTS AND DISCUSSION

Based on the above results, the total force can be obtained by performing integration of the force components over the surface of a sphere. To do this, we need to transform all of the above formulae back to the global coordinate system. According to the the geometry shown in Fig. 2,

$$F_X = \int_0^{2\pi} \int_0^{\pi} \{p \sin\theta \cos\phi + \tau_x \sin\phi + \tau_z \cos\theta \cos\phi\} R_0^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \tag{18}$$

Using the results for p, τ_x, τ_z obtained above in Eq. (18), we, after lengthy integration, obtain the drag force in the X direction as

$$F_X = \frac{1}{2}\pi R_0^2 \rho U_{f0}^2 \left\{ \frac{2-\sigma_p+\sigma_\tau}{2S^3} \left[\frac{4S^4-4S^2-1}{2S} \mathrm{erf}(S) + \frac{2S^2+1}{\sqrt{\pi}} \mathrm{e}^{-S^2} \right] + \frac{2\sigma_p}{3S}\sqrt{\pi} \right\} \tag{19}$$

where $S = U_{f0}/\sqrt{2RT}$. In this and the following integrations,we make wide use of the relation

$$\int_0^{2\pi} \mathrm{erf}(b\cos\phi)\cos\phi \mathrm{d}\phi = \frac{2b}{\sqrt{\pi}} \int_0^{2\pi} \mathrm{e}^{-b^2\cos^2\phi} \sin^2\phi \mathrm{d}\phi \tag{20}$$

where b is any function independent of ϕ.

The parameters of rotation and shear flow, i.e. Ω and G do not appear in Eq. (19), thus the drag force is not affected by the rotation of the sphere or the gradient of the shear flow. Equation (19) is actually the same as what would be obtained if the sphere were fixed in a uniform flow of a highly rarefied gas at the speed U_{f0}.

Similarly, we can calculate the lift force along the Y and Z directions and the results are

$$F_Z = -\frac{2}{3}\sigma_\tau \pi \rho \Omega R_0^3 U_{f0} \tag{21}$$

$$F_Y = -\frac{1}{6}(2+\sigma_\tau-\sigma_p)\pi\rho G R_0^2 \lambda U_{f0} \tag{22}$$

In Eq.(22), we only retain the term linear in U_{f0} since higher order terms involve βU_{f0} which is much smaller than 1 in our case.

The lift force in the Z direction involves only the rotation parameter Ω while that in the Y direction involves only G. Thus the rotation effect and shear flow effect are decoupled.

They do not produce any coupled effects. Equation (21) is the same as that derived by Wang [11] for the case when the fluid flow is uniform.

Equation (22) gives the lift force in the Y direction. This force is in the opposite direction from the Saffman force, Eq. (3), for the case when the fluid is continuum. Comparison of Eq. (22) and Eq. (3) shows that they have the same dependence on the velocity of the center of the sphere relative to the fluid flow and the radius of the sphere. However, the dependence on the gradient of the shear flow is different. The force is proportional to $\sqrt{G}$ when the rarefaction effect is absent but becomes linear in G when the fluid is highly rarefied. At first sight, Eq. (22) appears to be independent of viscosity. But according to the kinetic theory, the viscosity is proportional to $\rho\lambda$. Thus F_Y is linearly proportional to the viscosity μ in contrast to the Saffman force where the force is proportional to $\sqrt{\mu}$. The Saffman force also depends on $\sqrt{\rho}$ which is absent in Eq. (22).

Since the particle's size R_0 is usually quite small, F_Z, which depends on R_0^3, becomes less important then F_Y, which is proportional to R_0^2. This is the same as in the continuum case where the Saffman force is usually much more important than the Magnus force when the Reynolds number of the flow is low [1].

5 CONCLUSION

In this paper we derived the expression for the forces on a rotating spherical particle moving in a shear flow of a highly rarefied gas. Since the gas is highly rarefied, the molecules observe the Chapman-Enskog distribution function, which is used to calculate the total force on the particle based on the kinetic theory. Analytic expressions are obtained for the force on a spherical particle for the special case where the gradient of the shear flow is along the same direction as the axis of the particle's rotation. The drag force turns out to be the same as if the particle were fixed and the flow were uniform. The lift forces in the Y and Z

directions are in directions opposite to their counter parts in the continuum case. These results are consistent with previous studies [11] [17] and are considered appropriate to be used to calculate the particle motion in a highly rarefied gas.

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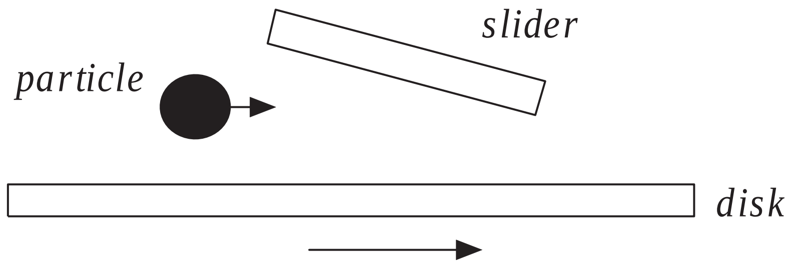

Figure 1: A particle moves into the head disk interface. The figure is not to scale

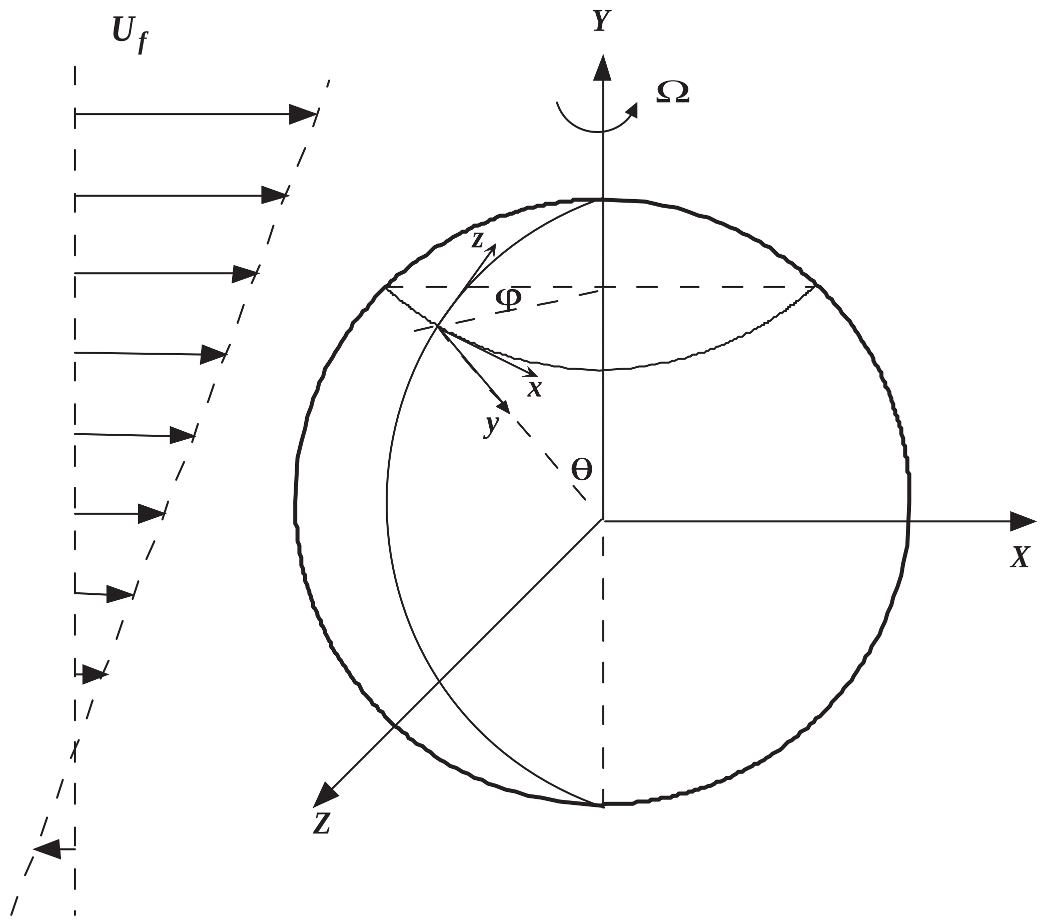

Figure 2: The global and local coordinate systems